# FloGARCH : Realizing long memory and asymmetries in returns volatility


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by Harry Vander Elst

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## Abstract

We introduce the class of FloGARCH models in this paper. FloGARCH models provide a parsimonious joint model for low frequency returns and realized measures and are sufficiently flexible to capture long memory as well as asymmetries related to leverage effects. We analyze the performances of the models in a realistic numerical study and on the basis of a data set composed of 65 equities. Using more than 10 years of high-frequency transactions, we document significant statistical gains related to the FloGARCH models in terms of in-sample fit, out-of-sample fit and forecasting accuracy compared to classical and Realized GARCH models.

JEL classification: C22, C53, C58, G17

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# 1 Introduction

Strong regularities in financial time series suggest that asset returns volatility is subject to temporal variation. Scholars in the field spurred intensive research in modeling the latent volatility process of asset returns. Among the existing approaches, conditional heteroskedastic models, pioneered by Engle (1982) and Bollerslev (1986) with the ARCH and GARCH models, have known undeniable success. Although originally designed for inflation modeling, ARCH models have been found to replicate stylized facts of asset returns highlighted by Mandelbrot (1963) including, but not limited to, volatility clustering, fat tails in the distribution of returns and higher-order dependence in returns. Standard models have been, since then, improved in three major directions; dealing with asymmetries, accommodating for long-range dependencies and exploiting the potential of high-frequency data. This paper makes a contribution at the intersection of these three axes by introducing a new class of long-memory asymmetric GARCH models based on high-frequency data. The three next paragraphs summarize recent developments on these three aspects.

First, standard extensions of the baseline models provide sufficient flexibility to capture the asymmetric relationship between returns and volatility documented in Black (1976). Notable contributions in this direction include among others the Exponential GARCH of Nelson (1991), the GJR-GARCH of Glosten *et al.* (1993), the asymmetric GARCH of Engle and Ng (1993), the Threshold GARCH of Zakoian (1994), the quadratic GARCH of Sentana (1995) and the family of smooth transition GARCH studied in González-Rivera (1998) and Anderson *et al.* (1999). Parameters constraints imposed to ensure positivity of the volatilities were also relaxed in some of these works (e.g. Nelson (1991)).

Second, another property found in financial returns is the long-range dependencies observed in squared and absolute returns. Long-memory properties are best reproduced by the hyperbolic rates of decay in the autocorrelation functions (henceforth ACF). Following Brockwell and Davis (1991), a covariance stationary process has a long memory if its ACF, $\rho(\cdot)$, is such that $\rho(k) \sim Ck^{2d-1}$ as $k \to \infty$ for $C > 0$ and $d < 0.5$. The first model to account for this property is the Integrated GARCH of Engle and Bollerslev (1986). Further contributions include fractionally integrated models such as the FIGARCH of Baillie *et al.* (1996), the FIEGARCH of Bollerslev and Mikkelsen (1996), the FIAPARCH of Tse (1998), the HYGARCH of Davidson (2004) and the Seasonal FIEGARCH of Lopes and Prass (2013). Diebold and Inoue (2001) argued that GARCH models with regime switches may also produce long-memory effects, which are not to be confused with those produced by fractionally integrated models. Other models include the Component GARCH model of Engle and Lee (1999) and the HARCH of Müller *et al.* (1997).

Third, all of the aforementioned models rely on an information set $\mathcal{F}_t = \sigma(r_t, r_{t-1}, \ldots)$ spanned by low frequency returns. However, the growing availability of high-frequency data has paved the way for a new type of volatility estimates, commonly known as realized measures, and defined as non-parametric estimators of the ex-post volatility of an asset over a fixed horizon (e.g. one day). The baseline realized variances were introduced in Andersen *et al.* (2001), and followed by many alternative estimators with different properties (discussed in Section 4). As illustrated on the top panel of Figure 1, realized measures provide a far more informative signal about the true latent volatility process than low frequency returns and extend the information set $\mathcal{F}_t = \sigma\left(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots\right)$ where $\mathcal{X}_t = \left(r_t, x_{t,1}, x_{t,2}, ..., x_{t,m}\right)'$ contains the low frequency return and $m$ different realized volatility measures. Not surprisingly, GARCH models relying on an extended information set have proven to provide significant economic and statistical gains and to react more quickly to sudden changes in the conditional volatility than their low frequency peers (see e.g. Christoffersen *et al.* (2012) and Andersen *et al.* (2003)).

Models including realized measures in the GARCH equation (i.e. GARCH-X) were introduced by Engle (2002) and further studied by Visser (2010). Hansen *et al.* (2012) completed GARCH-X models with a measurement equation for the realized measure leading to the class of Realized GARCH models. Later, Hansen and Huang (2012) introduced the Realized EGARCH to account for leverage effects and Hansen *et al.* (2014b) the multivariate Realized Beta GARCH. Competing models include the multiplicative error model (MEM) of Engle and Gallo (2006) and the HEAVY model of Shephard and Sheppard (2010). Further models were constructed to directly forecast the realized measures instead of the conditional variance of returns and include ARFIMA models (Andersen *et al.* (2003)), long-memory factor models (Luciani and Veredas (2015)) and the well-known HAR-RV models (Corsi (2009)). They are of particular interest in this paper as they all accommodate long-range dependencies in realized measures (see Andersen *et al.* (2003)) and will be part of the set of competing models in the section devoted to forecasting. Further high-dimensional semi-parametric approaches include Barigozzi *et al.* (2014).

This paper introduces a new class of volatility models belonging to the class of Realized GARCH introduced by Hansen *et al.* (2012). Classical Realized GARCH fail to reproduce long-range dependencies in the ACF of the realized measure. On the bottom panel of Figure 1, both solid lines represent the ACF of realized kernels estimated from a S&P 500 ETF. [1] On the left side, the bars provide the ACF of realized measures simulated from the Realized GARCH of Hansen *et al.* (2012) and, on the right side, the realized measures simulated from our new long-memory model. The level of decay in the bars on the left panel is faster than the solid line suggesting that the Realized GARCH model does not capture the persistence found in the estimated realized kernels. This empirical feature motivates the introduction of long-memory Realized GARCH. The right panel shows that both the bars and the solid line decay at the same pace, which provides evidence on the empirical usefulness long-memory models. The new subclass of Realized GARCH is called FloGARCH standing for **f**ractiona**l**ly integrated realized v**o**latility GARCH. The novelty of FloGARCH models lies in the combination of fractionally integrated polynomials for long memory, leverage functions for asymmetries and the use of high-frequency data, which results in a flexible and parsimonious class of models. This paper documents substantial improvements for modeling volatilities that can be gained from the use of our models. A realistic numerical experiment sheds light on the in-sample properties of the quasi-maximum likelihood estimation procedure and on the parameters' stability. Extensive estimation results are provided along with numerous empirical findings. We also test

[1] Standard & Poors Depository Receipt – SPY henceforth.

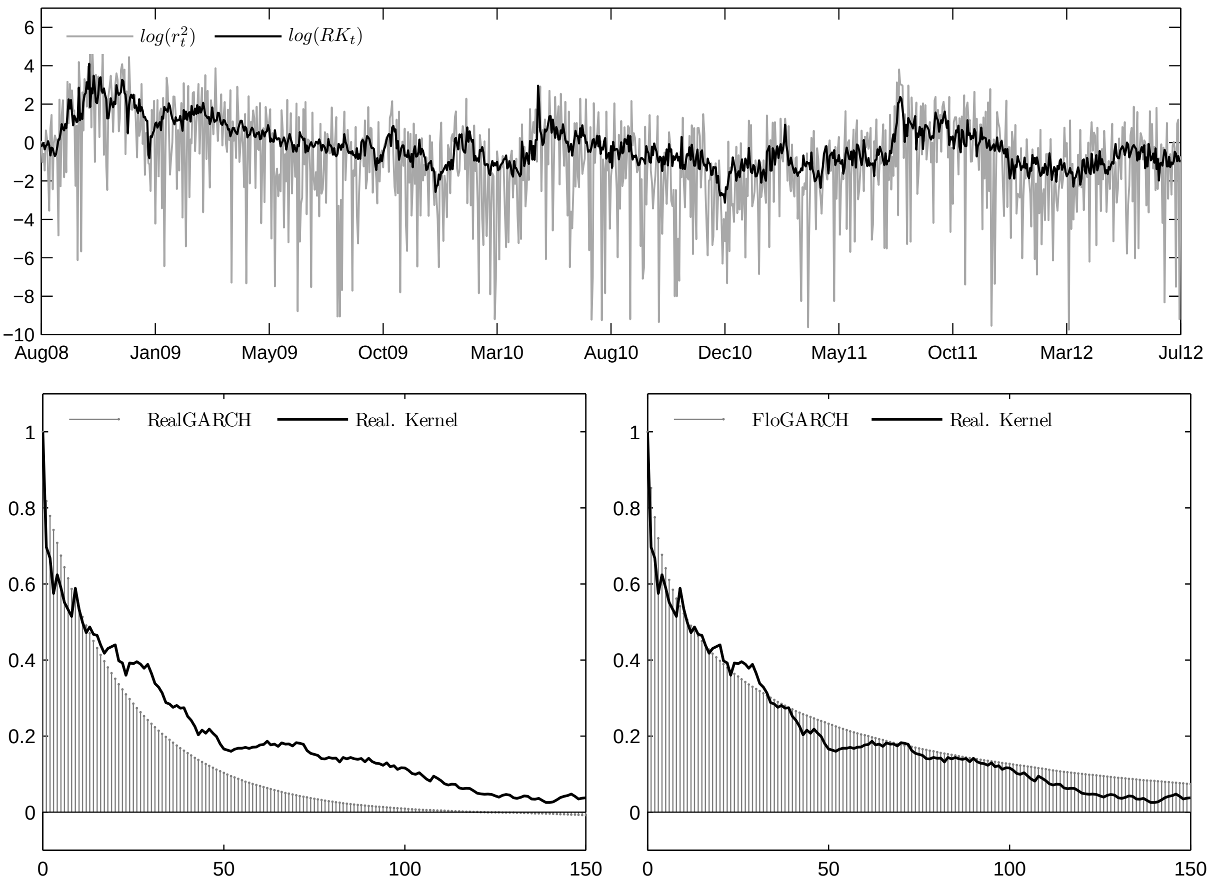


Figure 1: The **top** panel of this figure plots the logarithm of squared returns against the logarithm of realized kernels and illustrates the relative noisiness of the signal provided by squared returns. The **bottom** panel provides the ACF of realized kernels against the ACF of realized measures simulated from the Realized GARCH model of Hansen *et al.* (2012) (left) and from the FloGARCH (right).

several likelihood functions and document the optimal implementation of FloGARCH models in terms of parameters restrictions and realized measures choice. In-and-out of sample likelihood metrics are provided for several realized measures and compared across all the available stocks. Finally, forecasting performances are reported and compared with competing long-memory models.

Throughout the paper, we use the following notation, unless explicitly stated otherwise: $r_t$ denotes the log-return at time $t$, $h_t$ denotes the conditional variance of returns at time $t$, $\breve{h}_t$ can denote either $h_t$ or $\log h_t$ depending on the model considered. For example, for a GARCH model, it denotes $h_t$ while it represents $\log h_t$ in the case of a LGARCH or EGARCH model. Finally, $x_t$ stands for the realized measure computed at at time $t$ and, L denotes the lag operator defined such that $LX_t = X_{t-1}$.

The paper proceeds as follows: Section 2 introduces notation and the FloGARCH models. In Section 3, the likelihood equations are provided and the quasi-maximum likelihood estimation procedure is discussed. Simulation and bootstrap results are also analyzed. Empirical results are located in Section 4. Section 5 presents forecasting results and Section 6 concludes. Additional results are reported in the Appendix.

## 2 FloGARCH models

This section provides a detailed presentation of three FloGARCH models and introduces the notations for the rest of the paper. FloGARCH models form a subclass of the general class of Realized GARCH models defined in Hansen *et al.* (2012) as

$$r_t = \mu + h_t^{1/2} z_t, \tag{1}$$

$$h_t = v(h_{t-1}, ...; x_{t-1}...; r_{t-1}, ...), \tag{2}$$

$$x_t = m(h_t, z_t, u_t), \tag{3}$$

where $z_t \sim i.i.d.\,(0,1)$ and $u_t \sim i.i.d.\,\left(0, \sigma_u^2\right)$ are two independent random variables. We label equation 1 as the return equation, equation 2 as the GARCH equation and equation 3 as the measurement equation. Moreover, $E[r_t|\mathcal{F}_{t-1}] = \mu$, $V[r_t|\mathcal{F}_{t-1}] = h_t > 0$ and $\mathcal{F}_t = \sigma\left(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots\right)$ with $\mathcal{X}_t = (r_t, x_t)'$. The conditional mean process is kept constant throughout this paper and we limit the amount of realized measures to one. Finally, our framework allows to integrate low-frequency squared returns in the GARCH equation. However, we follow empirical findings of Hansen *et al.* (2012), who showed that low-frequency returns were not informative in the presence of realized measures, and do not include daily returns for the sake of clarity. The rest of Section 2 is divided into two parts linking the FloGARCH models with their low-frequency counterparts.

### 2.1 Linear FloGARCH and FloLGARCH

The Realized GARCH(p,q) and Realized LGARCH(p,q) of Hansen *et al.* (2012) can be written as

$$r_t = \mu + h_t^{1/2} z_t,$$

$$\breve{h}_t = \bar{\omega} + \beta(L)\breve{h}_t + \alpha(L)\breve{x}_t,$$

$$\breve{x}_t = \xi + \varphi\breve{h}_t + \delta(z_t) + u_t,$$

where $\beta(L) = \beta_1 L + \cdots + \beta_p L^p$, $\alpha(L) = \alpha_1 L + \cdots + \alpha_q L^q$, $\breve{h}_t$ and $\breve{x}_t$ denote either $h_t$ and $x_t$ or their logarithmic transformations and $\delta(z_t) = \delta_1 z_t + \delta_2\left(z_t^2 - 1\right)$ captures the leverage effect in the measurement equation. [2] The modeling strategy for the return and the realized measure is identical for all FloGARCH models. The main input is provided in the GARCH equation. Following the construction of the FIGARCH introduced by Baillie *et al.* (1996), the GARCH equation is transformed to include long-memory effects. If $\nu_t = \breve{x}_t - \breve{h}_t$, the ARMA representation of the model is given by

$$(1 - \alpha(L) - \beta(L))\breve{x}_t = \bar{\omega} + (1 - \beta(L))\nu_t.$$

Similarly to GARCH models, the estimated polynomial $1 - \hat{\alpha}(z) - \hat{\beta}(z) = 0$ is typically found to have roots close to 1 suggesting that $\breve{x}_t$ may be an I(1) process. [3] However, a large strand of the literature has

[2] In the FloLGARCH model, we have that $h_t = exp(\breve{h}_t)$.

[3] More empirical evidence about the persistence parameter of Realized GARCH models can be found in Table 2 of Hansen *et al.* (2012).

underlined the mean reverting property of volatility and suggested that fractional orders of integration may reconcile both stylized facts. As pointed out by Baillie *et al.* (1996) and Bollerslev and Mikkelsen (1996), factorizing the autoregressive polynomial $(1-\alpha(z)-\beta(z)) = \gamma(L)(1-z)^d$, where $\gamma(z)=0$ has roots outside the unit circle, allows for long-range dependencies in $\check{x}_t$. The model can then be written as

$$\check{h}_t = \omega + \left[1-\gamma(L)(1-\beta(L))^{-1}(1-L)^d\right]\check{x}_t. \tag{4}$$

The fractional differencing operator $(1-L)^d$ is defined by its Maclaurin series expansion. Denoting the gamma function by $\Gamma(\cdot)$, one obtains,

$$(1-L)^d = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}.$$

The volatility process of the linear FloGARCH and the FloLGARCH models is defined by equation 4. Both of them can be seen as the high-frequency counterparts of the FIGARCH and the FILGARCH models respectively and, for the sake of compactness, can be written using their *Realized ARCH(∞)* form

$$\check{h}_t = \omega + \lambda(L)\check{x}_t,$$

where $\lambda(L) = \sum_{j=0}^{\infty} \lambda_j L^j$ and $\lambda_0 = 0$ (see Appendix 7.1 for more details on the computation of the coefficients). The model implies a long memory structure on $r_t^2$ and $\check{x}_t$ through the GARCH equation. The Flo(L)GARCH(1,d,1) specification can be written as $\check{h}_t = \omega + \left[1-(1-\gamma L)(1-\beta L)^{-1}(1-L)^d\right]\check{x}_t$ and will be used our empirical application.

Importantly, Baillie *et al.* (1996) showed that the FIGARCH model is not weakly stationary for $0 < d < 1$. By contrast, following results from Nelson (1990), they pointed out that, under some conditions, the FIGARCH model is strictly stationary and ergodic. Many questions concerning weakly stationary solutions remain open for the FIGARCH. In contrast, FloGARCH models are based on $\check{x}_t$ and not on low frequency returns. In fact, stationary solutions found in the case of low frequency models do not necessarily hold for Realized GARCH models. There is a wide literature on ARCH(∞) stationary processes (see e.g. Kazakevicius and Leipus (2002), Zaffaroni (2004) and Giraitis *et al.* (2009)) and extending the existing results to the case of the Realized ARCH(∞) is left for future research.

## 2.2 FloEGARCH

The construction of the FloEGARCH is inspired from the FIEGARCH model of Bollerslev and Mikkelsen (1996). The Realized EGARCH(1,1) was introduced by Hansen and Huang (2012). A more general specification for Realized EGARCH(p,q), following from the definition of the EGARCH(p,q) given by Nelson (1991), includes several lags of the leverage function $\tau(z_t)$. The starting point is the equation of the logarithmic volatility

$$\begin{aligned}
r_t &= \mu + e^{\check{h}_t/2} z_t, \\
\check{h}_t &= \bar{\omega} + \vartheta(L)\check{h}_t + \alpha(L)\tau(z_t) + \gamma(L)u_t, \\
\check{x}_t &= \xi + \varphi\check{h}_t + \delta(z_t) + u_t,
\end{aligned}$$

where $\breve{h}_t = log(h_t)$ and $\breve{x}_t = log(x_t)$. Additionally $\vartheta(L) = \vartheta_1 L + \cdots + \vartheta_p L^p$, $\alpha(L) = \alpha_1 L + \cdots + \alpha_q L^q$ and $\gamma(L) = \gamma_1 L + \cdots + \gamma_q L^q$. The polynomial accounting for leverage effects is often written with $\alpha_1 = 1$. This definition departs from the usual EGARCH(p,q) model not only through the inclusion of realized measures, but also in the form of the news impact function. Originally specified as $\tau(z_t) = \tau_1 z_t + \tau_2(|z_t| - E|z_t|)$, it will be parametrized here as an Hermite polynomial of degree 2, $\tau(z_t) = \tau_1 z_t + \tau_2(z_t^2 - 1)$. Bollerslev and Mikkelsen (1996) underlined that the estimated polynomial $\hat{\vartheta}(z) = 1$ often presents roots close to one. Tables 2 to 4 from Hansen and Huang (2012) confirm the validity of this empirical feature for the Realized EGARCH, which motivates the FloEGARCH. Factorizing $1 - \vartheta(z) = \beta(z)(1-z)^d$, where all the roots of $\beta(z) = 0$ lie outside the unit circle, the GARCH equation of the FloEGARCH model may be written as

$$\breve{h}_t = \omega + \beta(L)^{-1}(1-L)^{-d}\Big[\alpha(L)\tau(z_t) + \gamma(L)u_t\Big],$$

where $\beta(z) = 1 - \sum_{i=1}^{p} \beta_i z^i$. The FloEGARCH(1,d,1) will be used in this paper and is given by

$$\breve{h}_t = \omega + (1-\beta L)^{-1}(1-L)^{-d}\Big[\tau(z_{t-1}) + \gamma u_{t-1}\Big], \tag{5}$$

where $\tau_1$ and $\tau_2$ capture the leverage effect and are usually negative and positive respectively. Likewise, $\delta(z_t)$ is also an Hermite polynomial of degree 2 and $\delta_1$ and $\delta_2$ exhibit equivalent signs as $\tau_1$ and $\tau_2$ respectively.

The FloEGARCH with low frequency returns has a strong connection with the EGARCH, the FIEGARCH and the Realized EGARCH models, which will be the main competing models among GARCH-type models. More details about the coefficients are provided in Appendix 7.2. For $d < 0.5$, Lopes and Prass (2014) proved that the FIEGARCH is weakly and strictly stationary under some further conditions extensively discussed in their work.

**Remark 1**. Temporal aggregation is an important challenge in time series models. A baseline example illustrates the difficulties related to high-frequency data. Consider the classical realized volatilities of Andersen *et al.* (2001)

$$E\Big[x_t|\mathcal{F}_{t-1}\Big] = E\Big[\sum_{i}^{n} r_{i,t}^2|\mathcal{F}_{t-1}\Big] = h_t,$$

where we assume that $r_{i,t} = \left(\frac{h_t}{n}\right)^{\frac{1}{2}}\epsilon_{i,t}$ with $\epsilon_{i,t} \sim i.i.d.(0,1)$. This model corresponds to a diffusion model with constant volatility and provides an intuitive way to link high-frequency returns to conditional daily volatilities. Nonetheless, three main challenges prevent the use of this approach. First intraday volatility is not constant. Second, intraday prices display large unexpected movements or jumps, which hampers proper measurement of volatility. Finally, it well-known that high-frequency data are polluted by microstructure noise. Hence, there is strong empirical evidence against this simple model and a more sophisticated approach should be used. However, this is beyond the scope of this paper and the three previous arguments motivate us to consider realized measures $x_t$ as exogenous signals and to provide them with a proper measurement equation that allows for simulations and multi-step ahead forecasts.

# 3 Quasi-maximum likelihood analysis

In this section, we discuss the estimation of the parameters of the FloGARCH models. For each of the three models, the estimation relies on quasi-maximum likelihood (henceforth QMLE) and the in-sample properties of the estimated parameters are uncovered in a realistic numerical experiment. Estimation results based on skewed and fat-tailed distributions are left for the Section 4.

## 3.1 Quasi-maximum likelihood and partial likelihood equations

A classical question related to GARCH models concerns the choice of the probability distribution used in the estimation. Because the distribution of residuals is often difficult to characterize properly, the usual maximum likelihood procedure may be infeasible and alternative techniques have to be used. Bollerslev and Wooldridge (1992) proposed to use the quasi-maximum likelihood technique in order to estimate the parameters of GARCH-type models. They showed that a misspecified Gaussian log-likelihood provides consistent and asymptotically normal results. Lee and Hansen (1994) showed consistency and asymptotic normality for strictly stationary and ergodic GARCH(1,1) models. Jensen and Rahbek (2004) extended their results to the case where stationarity and ergodicity do not hold for the GARCH(1,1) process. Robinson *et al.* (2006) established the same results for ARCH(∞) processes under certain regularity conditions. QMLE is also the estimation procedure used by Engle (2002), Shephard and Sheppard (2010), Hansen *et al.* (2012) and Hansen and Huang (2012). Hence, there is strong evidence in favor of QMLE in many instances including non-standard cases. Consistently with the literature, we use Gaussian-QMLE for the FloGARCH models and document the goodness of the procedure in a numerical study based on the parametric bootstrap (see Paparoditis and Politis (2009)). Conditionally on $\mathcal{F}_t = \sigma(\chi_s, s \leq t)$ where $\chi_s = \{r_s, \mathrm{x}_s\}$, the log-likelihood function can be recursively separated as

$$l(r, x; \boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(r_t, x_t; \theta | \mathcal{F}_{t-1}),$$

which provides the objective function to maximize. Using Bayes' rule, we have the decomposition $f(r_t, x_t | \mathcal{F}_{t-1}) = f(r_t | \mathcal{F}_{t-1}) f(x_t | r_t, \mathcal{F}_{t-1})$, which allows to extract the partial likelihood of the return equation from the joint likelihood of the model. It will be useful to compare Realized GARCH models with standard GARCH. Using the logarithmic transformation, the full objective function becomes

$$l(r, x; \boldsymbol{\theta}) = -\frac{1}{2} \sum_{t=1}^{T} [\log(2\pi) + \log(h_t) + (r_t - \mu)^2 / h_t] - \frac{1}{2} \sum_{t=1}^{T} [\log(2\pi) + \log(\sigma_u^2) + u_t^2 / \sigma_u^2].$$

The parameters contained in the vector $\boldsymbol{\theta}$ are estimated by maximizing the objective function. The FloGARCH and FloLGARCH models have the same parameters $\boldsymbol{\theta_L} = (\mu, \omega, \gamma, d, \boldsymbol{\beta}, \xi, \varphi, \delta_1, \delta_2, \sigma_u^2)$ while the FloEGARCH has more parameters $\boldsymbol{\theta_E} = (\mu, \omega, d, \boldsymbol{\beta}, \boldsymbol{\alpha}, \tau_1, \tau_2, \gamma, \xi, \varphi, \delta_1, \delta_2, \sigma_u^2)$. The first summand of the objective function denotes the partial log-likelihood of returns and stands as the objective function used to estimate GARCH models. Therefore, it is taken as the basis to compare different models. In-sample and out-of-sample likelihood are used in the analysis of the models: Say, a sample of size $N$ is available and we decide to divide it in two subsamples. We use the first subsample to estimate the model. The likelihood resulting from the estimation is referred to as the in-sample likelihood. Then, using the estimated parameters, we compute the likelihood of the second subsample and call it the out-of-sample likelihood. The latter provides a measure of out-of-sample fit. Properties of these quantities and related statistics are studied in Hansen (2009).

The polynomial in the ARCH($\infty$) representation needs to be truncated for estimation. Baillie *et al.* (1996) and Bollerslev and Mikkelsen (1996) showed that using a truncation level of 1000 lags provides good in-sample results without destroying the long-term structure. The optimal level of truncation for FloGARCH models is analyzed on our panel of stocks by comparing the in-sample likelihood at different levels of truncation. Figure 2 provides the average likelihood and the 0.25 and 0.75 percentiles for different levels of truncation and results indicate that the truncation level has little impact on the likelihood for the FloGARCH and the FloLGARCH. However, the likelihood of the FloEGARCH increases with the truncation level, advocating in favor of higher levels of truncation.

A further complication arises from the treatment of initial conditions. Several initial conditions for FloGARCH and FloLGARCH models have been tested and provided fairly similar results in the estimation. Accordingly, the first observation of the (logarithmic) realized measure is used as starting point for the volatility filter. This departs from Baillie *et al.* (1996) who used the unconditional variance estimator. We follow Bollerslev and Mikkelsen (1996) for the FloEGARCH and set the initial value of the leverage functions to zero.

## 3.2 Numerical studies

A parametric bootstrap procedure based on Paparoditis and Politis (2009) is used to investigate the in-sample properties of the estimated parameters and confirms the validity of QMLE. Figure 3 represents the empirical standardized distribution of the estimated parameters computed with kernel densities. Each column provides results for one model: first, the linear FloGARCH (FloLin), second, the FloLGARCH (FloLog) and then the FloEGARCH (FloExp). Each row provides the kernel densities for the parameter with reference on each plot. The three FloGARCH models specified as (1,d,1) have been estimated on basis of the daily returns and realized measures of SPY. Then, residuals have been used to re-sample the observations and re-estimate the models several times. The re-sampling procedure is based on 10.000 samples of size $T = 1000$.

Figure 3 suggests that most of the estimated parameters have an in-sample distribution similar to a standard normal distribution. Nonetheless, the peaks and the asymmetry in the densities of $\hat{d}$ and $\hat{\beta}$ point to a less standard distribution for the FloGARCH and the FloLGARCH. Moreover, outliers may be present on the left side of the density of $\hat{d}$. Results for FloEGARCH are more convincing as all the estimated empirical densities are very close to a standard normal distribution. Finally, standard errors computed from the bootstrap procedure provide very similar results to the robust standard errors of parameters based on the usual sandwich-formula. Only $\hat{\mu}$ provides different results (of order $O(100)$) for

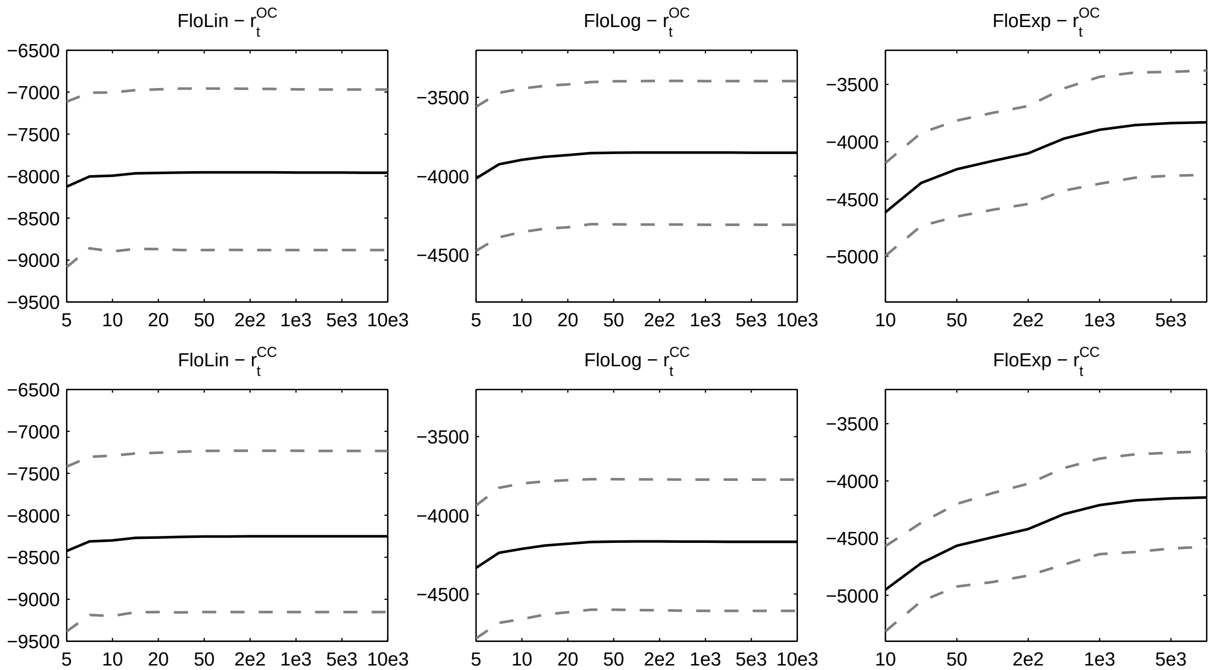


Figure 2: This figure shows the likelihood as a function of the truncation level both for open-to-close and close-to-close returns. The models considered are the FloGARCH(1,d,1), the FloLGARCH(1,d,1) and the FloEGARCH(1,d,1). The data base used for the estimation is described in the next section.

the standard deviation. Constrained versions of the model imposing $\mu = 0$ are discussed below.

# 4 Empirical analysis

Section 4 is based on data for 64 stocks and one ETF of the S&P 500 (SPY) traded on the NYSE from January 2002 to April 2012. High-frequency data was obtained from the TAQ database and cleaned following Barndorff-Nielsen *et al.* (2009). Open-to-close and close-to-close log-returns were computed for each stock. Realized measures rely on high-frequency data spanning the official market opening hours, i.e. between 9:30 am and 4:00 pm, but do not contain overnight information. Hence, they provide a precise proxy of the latent volatility of open-to-close returns but are downward biased signals of close-to-close returns' volatility. This bias will be reflected in the coefficients of the model. The sample is divided into two subsamples covering the periods from January 2002 to December 2008 (in-sample period) and from January 2009 to April 2012 (out-of-sample period) respectively. Within the data set, several trading days are removed to avoid outliers in the estimation. Incomplete trading sessions are omitted if recorded trades cover less than 90% of the trading time. Finally, zero returns are replaced by $10^{-5}$ in order to avoid issues related to the logarithmic transformations.

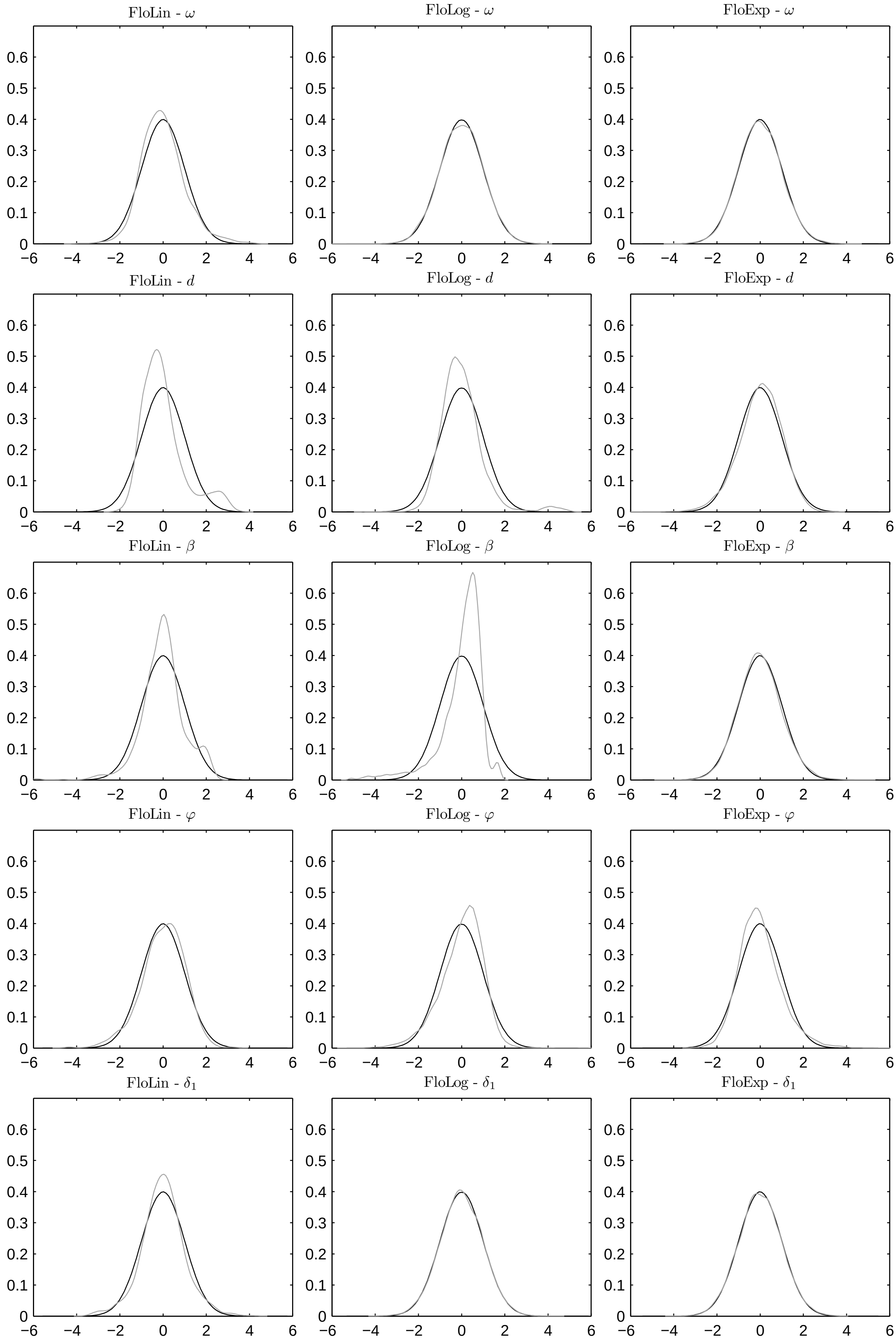


Figure 3: This figure provides the in-sample standardized distribution of some estimated parameters from the different FloGARCH models. The distributions were generated from a parametric bootstrap procedure on the basis of estimated residuals from SPY.

Table 1: Estimation results for SPY.

| Model | FloLin | | | | | | FloLog | | | | | | FloExp | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $(0,d,1)^{oc}$ | $(0,d,1)^{cc}$ | $(1,d,0)^{oc}$ | $(1,d,0)^{cc}$ | $(1,d,1)^{oc}$ | $(1,d,1)^{cc}$ | $(0,d,1)^{oc}$ | $(0,d,1)^{cc}$ | $(1,d,0)^{oc}$ | $(1,d,0)^{cc}$ | $(1,d,1)^{oc}$ | $(1,d,1)^{cc}$ | $(0,d,1)^{oc}$ | $(0,d,1)^{cc}$ | $(1,d,1)^{oc}$ | $(1,d,1)^{cc}$ | $(1,d,1)^{oc}_\star$ | $(1,d,1)^{cc}_\star$ |
| *Panel A: Point estimates* | | | | | | | | | | | | | | | | | | |
| $\mu$ | 0.00 (0.00) | 0.02 (0.01) | 0.00 (0.00) | 0.02 (0.01) | 0.00 (0.00) | 0.02 (0.01) | 0.00 (0.00) | 0.03 (0.01) | 0.00 (0.02) | 0.03 (0.01) | 0.00 (0.01) | 0.03 (0.01) | −0.00 (0.01) | 0.03 (0.01) | −0.00 (0.01) | 0.02 (0.01) | −0.00 (0.02) | 0.03 (0.01) |
| $\omega$ | 0.04 (0.01) | 0.24 (0.01) | 0.04 (0.01) | 0.24 (0.02) | 0.04 (0.01) | 0.24 (0.02) | 0.16 (0.02) | 0.54 (0.02) | 0.16 (0.02) | 0.54 (0.02) | 0.16 (0.02) | 0.54 (0.02) | −0.27 (0.28) | −0.04 (0.10) | −0.25 (0.21) | −0.01 (0.03) | −0.16 (0.24) | 0.05 (0.20) |
| $\gamma$ | 0.04 (0.03) | 0.02 (0.02) | | | 0.07 (0.04) | 0.02 (0.04) | −0.09 (0.02) | −0.10 (0.02) | | | 0.29 (0.07) | 0.28 (0.06) | 0.44 (0.02) | 0.41 (0.01) | 0.42 (0.02) | 0.39 (0.02) | 0.43 (0.02) | 0.40 (0.02) |
| $d$ | 0.75 (0.05) | 0.77 (0.04) | 0.78 (0.05) | 0.80 (0.05) | 0.79 (0.06) | 0.78 (0.06) | 0.66 (0.02) | 0.67 (0.02) | 0.67 (0.02) | 0.68 (0.02) | 0.75 (0.04) | 0.76 (0.04) | 0.73 (0.01) | 0.74 (0.01) | 0.68 (0.02) | 0.68 (0.01) | 0.68 (0.02) | 0.68 (0.01) |
| $\beta$ | | | 0.44 (0.06) | 0.45 (0.05) | 0.46 (0.08) | 0.41 (0.09) | | | 0.11 (0.03) | 0.12 (0.03) | 0.47 (0.09) | 0.47 (0.08) | | | 0.15 (0.03) | 0.16 (0.03) | 0.17 (0.03) | 0.17 (0.03) |
| $\tau_1$ | | | | | | | | | | | | | −0.12 (0.01) | −0.18 (0.01) | −0.11 (0.01) | −0.17 (0.01) | −0.12 (0.01) | −0.17 (0.01) |
| $\tau_2$ | | | | | | | | | | | | | 0.06 (0.00) | 0.03 (0.00) | 0.06 (0.00) | 0.03 (0.00) | 0.15 (0.01) | 0.09 (0.01) |
| $\xi$ | 0.04 (0.01) | −0.14 (0.04) | 0.04 (0.03) | −0.13 (0.03) | 0.04 (0.02) | −0.14 (0.04) | −0.17 (0.02) | −0.54 (0.02) | −0.17 (0.02) | −0.54 (0.02) | −0.17 (0.02) | −0.54 (0.02) | −0.16 (0.02) | −0.55 (0.02) | −0.16 (0.02) | −0.54 (0.02) | −0.17 (0.02) | −0.55 (0.02) |
| $\varphi$ | 0.93 (0.02) | 0.93 (0.02) | 0.93 (0.02) | 0.93 (0.01) | 0.93 (0.01) | 0.93 (0.02) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 0.98 (0.01) | 0.98 (0.01) | 1.04 (0.02) | 1.01 (0.02) | 1.05 (0.02) | 1.02 (0.02) | 1.02 (0.02) | 1.01 (0.02) |
| $\delta_1$ | −0.03 (0.01) | −0.18 (0.02) | −0.03 (0.02) | −0.18 (0.02) | −0.03 (0.02) | −0.18 (0.02) | −0.08 (0.01) | −0.14 (0.01) | −0.08 (0.01) | −0.14 (0.01) | −0.08 (0.00) | −0.14 (0.01) | −0.09 (0.01) | −0.15 (0.01) | −0.09 (0.01) | −0.15 (0.01) | −0.09 (0.01) | −0.15 (0.01) |
| $\delta_2$ | 0.10 (0.02) | 0.03 (0.01) | 0.10 (0.02) | 0.03 (0.01) | 0.10 (0.01) | 0.03 (0.01) | 0.06 (0.00) | 0.02 (0.00) | 0.06 (0.00) | 0.02 (0.00) | 0.06 (0.00) | 0.02 (0.00) | 0.06 (0.00) | 0.02 (0.00) | 0.06 (0.00) | 0.02 (0.00) | 0.06 (0.00) | 0.02 (0.00) |
| *Panel B: Log-likelihood and Residuals Variance* | | | | | | | | | | | | | | | | | | |
| $l(r,x)$ | −7641.5 | −8125.8 | −7641.4 | −8125.8 | −7641.4 | −8125.8 | −4133.8 | −4544.8 | −4133.4 | −4544.3 | −4131.8 | −4542.6 | −4055.3 | −4420.8 | −4049.4 | −4412.9 | −4052.3 | −4416.9 |
| $l(r)$ | −3096.6 | −3587.1 | −3096.3 | −3586.8 | −3096.2 | −3587.0 | −3095.5 | −3554.9 | −3095.2 | −3554.5 | −3094.3 | −3554.0 | −3101.0 | −3554.3 | −3102.2 | −3552.7 | −3102.0 | −3559.5 |
| $\hat{\sigma}_u^2$ | 2.39 | 2.37 | 2.39 | 2.37 | 2.39 | 2.37 | 0.14 | 0.13 | 0.14 | 0.13 | 0.14 | 0.13 | 0.13 | 0.12 | 0.13 | 0.12 | 0.13 | 0.12 |

Table 1 provides estimation results based on the full sample for SPY, i.e. from January 2002 to April 2012. Different specifications are analyzed from the three subclasses of models both for open-to-close and close-to-close returns. Both $(1,d,1)^{oc}_\star$ and $(1,d,1)^{cc}_\star$ denote a version of the FloEGARCH based on the leverage function $\tau(z_t) = \tau_1 z_t + \tau_2(|z_t| - E|z_t|)$ used in standard EGARCH models. Robust standard errors are provided in parenthesis and are numerically obtained from the scores and the Hessian matrix of the log-likelihood function.

**Remark 2**. Realized measures provide an imperfect signal for the conditional volatility and may be conditionally biased. Since $\delta(z_t) + u_t$ is a martingale difference sequence, we have from the measurement equation that

$$E\left[\breve{x}_t|\mathcal{F}_{t-1}\right] = \xi + \varphi E\left[\breve{h}_t|\mathcal{F}_{t-1}\right].$$

Realized measures would be conditionally unbiased if the parameters $\xi$ and $\varphi$ were be found to be equal to 0 and 1, respectively.

## 4.1 Estimation results

Estimation results are presented for SPY and summarized in Table 1. Further estimation results for the FloLGARCH and the FloEGARCH can be found in Appendix 7.3.

First, notable differences appear in the estimated mean parameter $\mu$ depending on the choice of returns, $r_t^{oc}$ or $r_t^{cc}$, and suggest that $\mu^{oc} = 0$ and $\mu^{cc} > 0$. This observation implies that $\mu^{co} > 0$ and leads to the conclusion that overnight information generates more performance for the market index. This is confirmed for each model estimated on SPY.

Second, the coefficients of the measurement equation are not sensitive to the specification of the GARCH equation and significantly different from zero. However, they are sensitive to the choice of returns and important differences appear in point estimates, e.g. $\xi^{oc} > \xi^{cc}$. These differences are explained by adjustments required to account for biases in realized measures with respect conditional volatilities. In fact, the biases in realized measures can be captured by the parameters $\xi$ and $\varphi$. If $\xi = 0$ and $\varphi = 1$, then $E[\breve{x}_t|\mathcal{F}_{t-1}] = \breve{h}_t$ and the realized measure is conditionally unbiased. The FloLGARCH and the FloEGARCH display stronger adjustments for open-to-close returns in the intercept, i.e. $\xi < 0$, which is also observed in Table 2 of Hansen *et al.* (2012). The FloLGARCH and the FloEGARCH display values for $\varphi$ close to one and the bias is corrected through a more negative intercept. In the linear model, $\xi^{oc}$ appears to be close to zero while the estimated values of $\varphi$ are strictly smaller than 1 and do not vary with the choice of returns.

### 4.1.1 Robustness check 1: A case against the linear specification

We discuss model validation and express our doubts about the linear specification. Figure 4 provides strong evidence against the linear specification. Similarly, Hansen *et al.* (2012) pointed out that the linear Realized GARCH provides odd results in terms of estimated residuals and displays high levels of heteroskedasticity. Figure 4 provides equivalent observations and leads to similar conclusions.

Three models have been estimated for SPY, the FloGARCH(1,d,1), the FloLGARCH(1,d,1) and the FloEGARCH(1,d,1). The two latter provide convincing results in terms of estimated residuals, as emphasized on the upper panel of Figure 4. The scatter plot of residuals $\{\hat{z}_t, \hat{u}_t\}$ is similar to the one of a bivariate standard normal distribution. Moreover, it appears that a plot of $\log x_t$ against the regressor $\log h_t$ provides evidence of homoskedasticity in the measurement equation. These facts are however

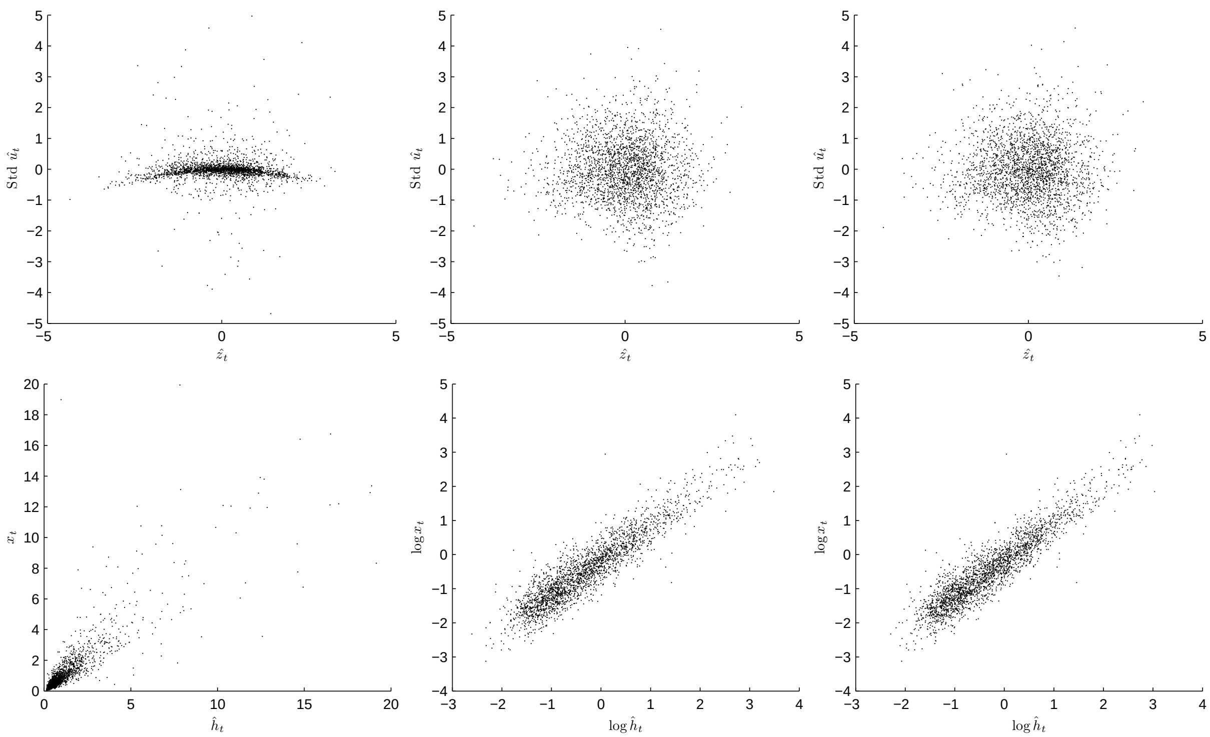


Figure 4: The upper panel of this figure provides a scatter plot of the estimated residuals $\{\hat{z_t}, \hat{u_t}\}$ respectively for the FloGARCH(1,d,1), the FloLGARCH(1,d,1) and the FloEGARCH(1,d,1) estimated on SPY. The x-axis provides values for $\hat{z}_t$ and the y-axis for $\hat{u}_t$. The lower panel provides scatter plots of $\hat{h}_t$ or $\log \hat{h}_t$ on the x-axis against $x_t$ or $\log x_t$ on the y-axis for the same models as on the upper panel.

transgressed by the linear FloGARCH. Residuals are not jointly normal and heteroskedasticity appears in the model validation plot. Hansen *et al.* (2012) suggested that a higher order leverage function may help to better capture heteroskedasticity, which decreases QMLE efficiency. As a result, we advocate in favor of logarithmic specifications through the FloLGARCH and FloEGARCH, which provide more convincing empirical results.

#### 4.1.2 Robustness check 2: introducing skewness and kurtosis

Alternative distributions are used to estimate the FloLGARCH and the FloEGARCH models. In the log-likelihood function

$$l(r, x; \boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(r_t|\mathcal{F}_{t-1}) + \sum_{t=1}^{T} \log f(x_t|r_t, \mathcal{F}_{t-1}),$$

we replace both $f(r_t|\mathcal{F}_{t-1})$ and $f(x_t|r_t, \mathcal{F}_{t-1})$ by the Student-t distribution and then by a skewed version of the Student-t distribution introduced by Fernández and Steel (1998). Both distributions have been used to estimate GARCH models by Bollerslev (1987) and Lambert *et al.* (2012). Results are useful to study the empirical distribution of the residuals. The Student-t distribution takes an additional parameter $\nu$ representing the degrees of freedom of the distribution, which accounts for the fat-tails found in residuals.

The higher this parameter, the closer residuals behave to a Gaussian distribution. The skewed version has two additional parameters $\nu$ and $\kappa$, which control tails and skewness respectively. The interpretation of $\nu$ is similar as before. Skewness follows this pattern: $\kappa = 1$ corresponds to a symmetric case, $\kappa > 1$ corresponds to a positive or right-skewed density while $\kappa < 1$ has a negative or left-skewed density. Table 2 provides the discussed parameters and their standard deviations. [4] All parameters are significant at 1% and suggest that residuals of both the return and the realized measure equations are not Gaussian. Fat-tails can be conjectured from the value $\nu$ and are more present for the residuals of the realized measure. Moreover, the tails of the returns residuals are fatter for close-to-close returns. Finally, skewness has opposite directions for the realized measures residuals and the returns residuals.

Table 2: Various distributions.

| | $FloLog^{oc}_{stud}$ | $FloLog^{cc}_{stud}$ | $FloLog^{oc}_{skew}$ | $FloLog^{cc}_{skew}$ | $FloExp^{oc}_{stud}$ | $FloExp^{cc}_{stud}$ | $FloExp^{oc}_{skew}$ | $FloExp^{cc}_{skew}$ |
|---|---|---|---|---|---|---|---|---|
| *Panel A: Point estimates for $z_t$:* | | | | | | | | |
| $\kappa_r$ | | | 0.84 (0.02) | 0.86 (0.02) | | | 0.84 (0.02) | 0.86 (0.02) |
| $\nu_r$ | 16.65 (4.22) | 8.95 (1.48) | 18.97 (6.52) | 9.30 (1.57) | 15.97 (4.79) | 8.47 (1.37) | 17.13 (5.54) | 8.92 (1.46) |
| *Panel B: Point estimates for $u_t$:* | | | | | | | | |
| $\kappa_x$ | | | 1.15 (0.03) | 1.14 (0.03) | | | 1.13 (0.03) | 1.11 (0.03) |
| $\nu_x$ | 7.59 (0.95) | 7.69 (0.98) | 7.81 (1.01) | 8.07 (1.08) | 7.49 (0.92) | 8.52 (1.19) | 7.69 (0.98) | 8.84 (1.29) |

Table 2 provides point estimates for the parameters $\nu$ and $\kappa$ when using a student or a skewed student to estimate the models' parameters. The FloLGARCH(1,d,1) and FloEGARCH(1,d,1) are tested with open-to-close and close-to-close returns. $z_t$ and $u_t$ denote the residuals of the return equation and measurement equation of the realized measure.

## 4.2 Realized measures comparisons

Subsection 4.2 documents that two dimensions in the realized measure choice improve the model performance. First, jump-robustness increases model fit and, second, subsampled and pre-averaged realized measures tend to provide more accurate signals of conditional variances.

Realized measures have been at the center of many recent developments at the frontier of probability theory and financial econometrics. [5] Realized volatilities are low frequency ex-post measures of volatility typically computed on a daily basis using high frequency data. The baseline realized variances introduced in the seminal work of Andersen *et al.* (2001), simply obtained by summing up the squared intraday returns, are known to produce upward biases in presence of market microstructure noise and jumps.

[4] We do not provide the parameters of the volatility models since they are similar to those obtained from the Gaussian-QMLE.

[5] See Barndorff-Nielsen and Shephard (2006), McAleer and Medeiros (2008) and Andersen and Benzoni (2008) for reviews on recent developments.

The impact of microstructure noise on realized variances (documented in Zhou (1996), Hansen and Lunde (2006) and Bandi and Russell (2008)) can be decreased by sparse sampling (Andersen *et al.* (2001)), subsampling (Zhang *et al.* (2005)), pre-averaging (Podolskij and Vetter (2009) and Jacod *et al.* (2009)) or by using realized kernels (Barndorff-Nielsen *et al.* (2008)).

Jumps represent an additional source of variation in asset returns, which hinders reliable measurements of the latent volatility. As a result, several jump-robust measures of volatility have been introduced including the bipower variation of Barndorff-Nielsen and Shephard (2004), the quantile-based realized variance of Christensen *et al.* (2010b) and the median and minimum realized variance of Andersen *et al.* (2012). Some extensions were proposed in the multivariate setup (see e.g. Vander Elst and Veredas (2014)).

Several recent realized measures are used to compare the ability of FloGARCH models to fit returns. Results for open-to-close and close-to-close returns are provided in Table 3 and in Table 9 in Appendix 7.3. These tables provide the sample mean of the parameters computed over the whole data set and summarize results for the (1,d,1) specification. Results are generally coherent between $r_t^{oc}$ and $r_t^{cc}$.

The pre-averaged quantile-based realized variance of Christensen *et al.* (2010b) is successful both in terms of in-sample and out-of-sample likelihood and provides the highest average level of memory for the linear specification. This finding is in line with similar evidence from Andersen *et al.* (2007) who reported the usefulness of the continuous component of the quadratic variation to describe assets volatility. Moreover, 1-min realized variances provide good in-sample fit for the FloLGARCH and the FloEGARCH. For $r_t^{oc}$, the 1-min subsampled MedRV of Andersen *et al.* (2012) is more successful than the baseline realized variance. This also true for $r_t^{cc}$ for the FloEGARCH but not for the FloLGARCH that shows higher out-of-sample fit when used with 1-min realized variances.

Two noteworthy remarks: First, both the in-sample and the out-of-sample likelihood decrease with the sampling frequency. Sparser frequencies decrease the amount of data used to compute realized measures and seems to worsen the model fit. Second, not surprisingly, subsampling increases the quality of both the in-sample and out-of-sample fit. Subsampling allows to smooth out the variability in realized measures produced by noisy log-returns and all the subsampled measures are more successful in terms of model fit than their standard versions.

## 4.3 Constrained estimation

Hansen and Huang (2012) pointed out that some parameters of the Realized EGARCH were very similar across assets. Moreover, they underlined that imposing restrictions to the models could improve the estimation procedure and make parameter interpretation easier. This subsection examines several constrained versions of FloGARCH models implemented with realized kernels and documents that, either no restrictions should be imposed or several parameters should be jointly constrained.

Restrictions are imposed on two parameters common to every model. First, $\mu = 0$ and $\varphi = 1$ are imposed separately in the various models and then both restrictions are imposed together. Restricted models are analyzed with $r_t^{oc}$ and $r_t^{cc}$. Results can be found in Table 4 and in Table 10 located in Appendix

Table 3: Comparison Table for different realized measures based on O-C returns.

| | Flo-Lin | | | | Flo-Log | | | | Flo-Exp | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ |
| $RV1m$ | 0.58 | 0.98 | −7773.75 | −3718.73 | 0.70 | 0.99 | −3628.80 | −1925.83 | 0.70 | 1.03 | −3623.89 | −1920.71 |
| $RV5m$ | 0.48 | 1.02 | −7700.19 | −3632.18 | 0.67 | 0.99 | −3913.25 | −2110.22 | 0.70 | 1.03 | −3900.57 | −2099.45 |
| $RV15m$ | 0.44 | 1.02 | −7764.72 | −3876.64 | 0.64 | 0.99 | −4266.78 | −2264.81 | 0.69 | 1.00 | −4240.29 | −2255.08 |
| $ssRV1m$ | 0.58 | 0.98 | −7773.75 | −3718.73 | 0.70 | 0.99 | −3628.80 | −1925.83 | 0.70 | 1.03 | −3623.89 | −1920.71 |
| $ssRV5m$ | 0.47 | 1.01 | −7731.18 | −3747.91 | 0.70 | 0.99 | −3845.47 | −2038.97 | 0.70 | 1.03 | −3819.59 | −2039.46 |
| $ssRV15m$ | 0.48 | 0.99 | −7419.73 | −3614.17 | 0.69 | 0.99 | −4096.69 | −2154.56 | 0.70 | 1.01 | −4069.30 | −2157.45 |
| $BPV1m$ | 0.59 | 0.98 | −7753.31 | −3708.64 | 0.66 | 1.00 | −3740.57 | −1999.80 | 0.70 | 1.05 | −3724.79 | −2010.98 |
| $BPV5m$ | 0.51 | 1.00 | −7596.86 | −3613.82 | 0.68 | 0.99 | −3964.14 | −2127.48 | 0.70 | 1.03 | −3958.05 | −2115.04 |
| $BPV15m$ | 0.48 | 1.01 | −7414.59 | −3870.88 | 0.66 | 0.99 | −4321.39 | −2316.67 | 0.70 | 1.01 | −4299.38 | −2303.98 |
| $MinRV1m$ | 0.60 | 0.98 | −7675.00 | −3736.27 | 0.69 | 1.00 | −3730.76 | −1984.46 | 0.70 | 1.02 | −3716.94 | −1982.24 |
| $MinRV5m$ | 0.49 | 1.01 | −7679.55 | −3630.45 | 0.68 | 0.99 | −4087.13 | −2194.72 | 0.70 | 1.02 | −4068.73 | −2183.45 |
| $MinRV15m$ | 0.46 | 1.02 | −7687.82 | −3963.74 | 0.65 | 0.99 | −4479.68 | −2412.75 | 0.70 | 1.02 | −4443.45 | −2399.14 |
| $ssMinRV1m$ | 0.61 | 0.97 | −7716.51 | −3616.88 | 0.69 | 1.00 | −3662.04 | −1930.25 | 0.70 | 1.03 | −3646.82 | −1928.52 |
| $ssMinRV5m$ | 0.53 | 0.99 | −7551.89 | −3655.44 | 0.69 | 0.99 | −3902.72 | −2076.56 | 0.70 | 1.03 | −3879.94 | −2082.53 |
| $ssMinRV15m$ | 0.49 | 1.00 | −7402.63 | −3687.39 | 0.67 | 0.99 | −4165.32 | −2234.36 | 0.70 | 1.02 | −4139.53 | −2224.02 |
| $prgMinRV$ | 0.52 | 1.01 | −9258.43 | −4386.10 | 0.53 | 1.04 | −3918.59 | −2130.35 | 0.70 | 1.19 | −3886.89 | −2108.23 |
| $MedRV1m$ | 0.60 | 0.97 | −7686.31 | −3665.40 | 0.69 | 1.00 | −3690.21 | −1927.49 | 0.71 | 1.03 | −3675.10 | −1924.91 |
| $MedRV5m$ | 0.51 | 1.01 | −7595.49 | −3629.39 | 0.69 | 0.99 | −3996.59 | −2153.39 | 0.70 | 1.03 | −3971.63 | −2137.56 |
| $MedRV15m$ | 0.48 | 1.00 | −7519.54 | −3862.12 | 0.64 | 0.99 | −4353.88 | −2340.53 | 0.70 | 1.02 | −4337.21 | −2328.09 |
| $ssMedRV1m$ | 0.60 | 0.97 | −7739.80 | −3604.07 | 0.69 | 0.99 | −3644.16 | −1905.74 | 0.70 | 1.04 | −3628.64 | −1903.22 |
| $ssMedRV5m$ | 0.53 | 0.98 | −7504.28 | −3650.60 | 0.70 | 0.99 | −3871.56 | −2064.78 | 0.70 | 1.03 | −3849.81 | −2060.61 |
| $ssMedRV15m$ | 0.53 | 0.99 | −7270.60 | −3608.36 | 0.66 | 0.99 | −4135.83 | −2209.87 | 0.70 | 1.03 | −4109.46 | −2205.10 |
| $prgMedRV$ | 0.45 | 1.06 | −9247.67 | −4405.88 | 0.54 | 1.03 | −3870.37 | −2096.90 | 0.70 | 1.14 | −3857.67 | −2074.83 |
| $RK$ | 0.56 | 1.00 | −7573.66 | −3625.53 | 0.69 | 0.99 | −3752.18 | −1959.25 | 0.70 | 1.03 | −3719.90 | −1960.90 |
| $TTS$ | 0.59 | 0.98 | −7426.43 | −3656.90 | 0.66 | 1.00 | −3723.39 | −2049.90 | 0.71 | 1.08 | −3718.42 | −2034.24 |
| $MSC$ | 0.59 | 0.99 | −7488.72 | −3599.30 | 0.67 | 1.00 | −3677.74 | −1920.79 | 0.71 | 1.09 | −3654.21 | −1938.58 |
| $prgRV$ | 0.51 | 1.00 | −7775.62 | −3683.42 | 0.70 | 0.99 | −3869.70 | −2022.98 | 0.69 | 1.03 | −3865.57 | −2027.86 |
| $prgQRV$ | 0.67 | 0.97 | −6951.38 | −3387.07 | 0.56 | 1.02 | −4824.43 | −2402.58 | 0.74 | 1.20 | −4717.99 | −2298.28 |
| $Average$ | 0.52 | 0.99 | −7677.27 | −3661.15 | 0.68 | 0.99 | −3887.14 | −2086.73 | 0.70 | 1.03 | −3872.76 | −2078.68 |

Table 3 provides estimation results for the FloGARCH(1,d,1), the FloLGARCH(1,d,1) and the FloEGARCH(1,d,1) estimated on the whole data set of stocks using different realized measures. The reported values correspond to the sample mean of each parameter computed over the results for the 65 securities. For each model, the value of the parameters $d$ and $\varphi$ are reported. The parameter $d$ summarizes the level of memory in the model while $\varphi$ is informative of potential biases contained in realized measures. The in-sample and the out-of sample likelihood was computed following the procedure described in Section 3. Results were obtained from $r_t^{oc}$. Equivalent results for $r_t^{cc}$ can be found in Appendix 7.3.

7.3. These tables report summary statistics for the in-sample likelihood, the the out-of-sample likelihood, the AIC and the BIC computed over the 65 assets of the data set.

Table 4: Comparison Table for constrained versions of FloGARCH - OC-returns.

| | FloLin | | | | FloLog | | | | FloExp | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ |
| *Panel A: Unrestricted Models* | | | | | | | | | | | | |
| *Average* | −7958.2 | −3618.5 | 15934.4 | 16049.4 | −3850.9 | −1948.4 | 7719.8 | 7834.8 | −3825.6 | −1943.2 | 7673.2 | 7813.8 |
| *Median* | −7573.7 | −3625.5 | 15165.3 | 15280.3 | −3752.2 | −1959.3 | 7522.4 | 7637.4 | −3719.9 | −1960.9 | 7461.8 | 7602.4 |
| $Q_{.25}$ | −8880.4 | −4250.0 | 13954.9 | 14069.9 | −4309.0 | −2173.9 | 6810.6 | 6925.6 | −4284.5 | −2173.9 | 6777.3 | 6917.8 |
| $Q_{.75}$ | −6968.4 | −2724.2 | 17778.9 | 17893.9 | −3396.3 | −1665.4 | 8636.1 | 8751.1 | −3377.6 | −1670.0 | 8591.0 | 8731.6 |
| *Panel B:* $\mu = 0$ | | | | | | | | | | | | |
| *Average* | −7959.7 | −3618.6 | 15935.3 | 16037.5 | −3852.4 | −1948.5 | 7720.7 | 7823.0 | −3827.2 | −1943.2 | 7674.3 | 7802.1 |
| *Median* | −7576.7 | −3625.3 | 15169.3 | 15271.6 | −3752.4 | −1960.9 | 7520.7 | 7623.0 | −3720.1 | −1960.2 | 7460.2 | 7588.0 |
| $Q_{.25}$ | −8882.0 | −4249.7 | 13953.1 | 14055.3 | −4309.7 | −2173.9 | 6808.8 | 6911.0 | −4284.9 | −2174.4 | 6775.2 | 6903.0 |
| $Q_{.75}$ | −6968.5 | −2725.5 | 17779.9 | 17882.1 | −3396.4 | −1664.8 | 8635.5 | 8737.7 | −3377.6 | −1670.0 | 8589.8 | 8717.6 |
| *Panel C:* $\varphi = 1$ | | | | | | | | | | | | |
| *Average* | −7960.8 | −3616.4 | 15937.7 | 16039.9 | −3852.2 | −1945.0 | 7720.4 | 7822.7 | −3828.7 | −1944.4 | 7677.4 | 7805.1 |
| *Median* | −7577.7 | −3624.0 | 15171.4 | 15273.6 | −3752.2 | −1959.2 | 7520.5 | 7622.7 | −3720.2 | −1961.2 | 7460.4 | 7588.2 |
| $Q_{.25}$ | −8881.7 | −4246.1 | 13969.7 | 14072.0 | −4309.2 | −2174.9 | 6809.0 | 6911.2 | −4286.9 | −2174.9 | 6776.3 | 6904.1 |
| $Q_{.75}$ | −6976.9 | −2726.4 | 17779.4 | 17881.6 | −3396.5 | −1664.9 | 8634.4 | 8736.6 | −3378.1 | −1669.6 | 8593.8 | 8721.6 |
| *Panel D:* $\mu = 0 \cap \varphi = 1$ | | | | | | | | | | | | |
| *Average* | −7962.3 | −3616.3 | 15938.5 | 16028.0 | −3853.5 | −1945.2 | 7721.0 | 7810.5 | −3830.1 | −1944.2 | 7678.3 | 7793.3 |
| *Median* | −7580.6 | −3623.8 | 15175.1 | 15264.6 | −3752.4 | −1960.8 | 7518.8 | 7608.3 | −3720.4 | −1960.5 | 7458.8 | 7573.8 |
| $Q_{.25}$ | −8883.7 | −4245.8 | 13969.8 | 14059.3 | −4309.9 | −2174.9 | 6807.1 | 6896.6 | −4287.8 | −2176.8 | 6774.4 | 6889.5 |
| $Q_{.75}$ | −6977.9 | −2727.5 | 17781.5 | 17870.9 | −3396.6 | −1664.2 | 8633.8 | 8723.3 | −3378.2 | −1669.2 | 8593.6 | 8708.6 |

Table 4 reports sample means of the statistics computed over the 65 stocks available in the data set. The results rely on $r_t^{oc}$ and the same specification is used for the three classes of models: (1,d,1). Results for $r_t^{cc}$ can be found in Appendix 7.3.

Results from Table 4 suggest that either no restriction should be imposed to the model or that $\mu = 0$ and $\varphi = 1$ should be imposed simultaneously. There is little evidence in favor of separate restrictions. The BIC criterion always points to smaller models and penalizes additional coefficients. Not surprisingly, the AIC, which tends to select larger models, often points to unrestricted models. Nevertheless, in some instances, it also provides evidence for smaller models and even never points to the unrestricted model for close-to-close returns. Consequently, these results advocate in favor of smaller models.

The in-sample likelihood of the unrestricted model is on average similar to restricted versions but is mainly choosing the unrestricted model for both types of returns. Statistical theory provides some intuition for this finding as bigger models often lead to better in-sample fit and weaker out-of-sample performances. In fact, the out-of-sample likelihood does not provide so clear results in our experiment. It picks often to restricted models but this is not always the case.

The main result is that parameters should be jointly constrained or left free. Yet, it is difficult to draw systematic conclusions from Table 4 and 10. Despite results found in Hansen and Huang (2012), our preferred specification remains, a priori, the unrestricted model. On the one hand, there is no unequivocal evidence in the presented tables pointing to one or the other restriction. On the other hand, these restrictions are heavily dependent on idiosyncratic properties of assets and more generally of asset classes. Nonetheless, those restrictions can be useful but should be tested for each asset separately.

### 4.4 Models comparison: in-sample and out-of-sample fit

In this section, several GARCH models are compared using in-sample and out-of-sample partial likelihood. Models are classified in three categories: linear, logarithmic and exponential. The linear class contains the baseline GARCH model and the FIGARCH extension. The high-frequency counterparts are represented by the Realized GARCH and the linear FloGARCH. The logarithmic class contains the LGARCH, the FILGARCH, the Realized LGARCH and the FloLGARCH while the exponential class is composed of the EGARCH, the FIEGARCH, the Realized EGARCH and the FloEGARCH.

Hansen and Huang (2012) have documented that the Realized EGARCH implemented with realized kernels provide the best fit. In order to provide a fair comparison basis, all models are estimated using realized kernels and the optimal implementation of FloGARCH provided in the previous subsection is ignored to avoid unfair comparison. Results are provided in Table 5 and contain summary statistics for the 65 securities.

There are three main observations. First, both for $r_t^{oc}$ and $r_t^{cc}$, adding a long memory component increases the average in-sample and out-of-sample fit of all models. Long term dependencies are found both in squared returns and realized measures and the statistical gains of long memory models confirm the need to account for it.

Second, as pointed out by many authors before, realized measures improve the models' performances and provide tangible statistical gains. Realized measures are far less noisy than returns and their gains have been documented in the literature (see Christoffersen *et al.* (2012)).

Third, the in-sample and out-of-sample partial likelihood evaluated on open-to-close returns provide better statistical fit than close-to-close returns. The reason is related to the data used to compute realized measures that only spans the trading day and does not contain overnight information. It should be mentioned that in most of the practical applications, close-to-close returns properties are of interest and adapting the realized measures to include close-to-open information may provide better statistical fit.

Finally, in many cases, FloGARCH models perform better than the competing models and provide better fit. The Realized GARCH models perform better in the linear and logarithmic category in terms of in-sample likelihood but only for close-to-close returns. The FloLGARCH and FloEGARCH provide the most convincing results for the models and outperform most of the competitors.

Table 5: Comparison Table for different GARCH models.

| | Linear | | | | Logarithm | | | | Exponential | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | GARCH | FIGARCH | R.GARCH | FloGARCH | LGARCH | FILGARCH | R.LGARCH | FloLGARCH | EGARCH | FIEGARCH | R.EGARCH | FloEGARCH |
| *Panel A: Open-to-close log-returns - In-sample partial likelihood* | | | | | | | | | | | | |
| *Average* | −3002.0 | −3000.7 | −2974.6 | −2974.5 | −3043.6 | −3158.4 | −2972.7 | −2971.3 | −2996.5 | −2994.2 | −3070.5 | −2966.7 |
| *Median* | −2912.9 | −2913.7 | −2925.9 | −2896.2 | −2952.1 | −3097.0 | −2888.8 | −2893.2 | −2905.0 | −2904.1 | −2934.4 | −2884.3 |
| $Q_{.25}$ | −3373.7 | −3372.4 | −3371.8 | −3366.4 | −3419.2 | −3498.9 | −3370.4 | −3368.6 | −3424.9 | −3374.6 | −3410.8 | −3365.9 |
| $Q_{.75}$ | −2662.8 | −2662.3 | −2629.5 | −2633.2 | −2703.6 | −2789.2 | −2629.8 | −2624.4 | −2653.3 | −2654.2 | −2628.7 | −2621.5 |
| *Min* | −4296.4 | −4293.0 | −4275.8 | −4270.5 | −4337.6 | −4340.6 | −4272.0 | −4267.3 | −4294.9 | −4293.9 | −7818.3 | −4265.2 |
| *Max* | −2096.0 | −2092.9 | −2036.2 | −2038.9 | −2114.3 | −2262.9 | −2039.3 | −2036.7 | −2071.2 | −2070.3 | −2043.2 | −2040.7 |
| % ≤ | 95.4 | 93.8 | 46.2 | 100.0 | 98.5 | 98.5 | 64.6 | 100.0 | 96.9 | 96.9 | 83.1 | 100.0 |
| *Panel B: Open-to-close log-returns - Out-of-sample partial likelihood* | | | | | | | | | | | | |
| *Average* | −1561.9 | −1560.1 | −1552.9 | −1549.5 | −1965.9 | −1627.9 | −1547.0 | −1545.4 | −1545.0 | −1560.6 | −1605.2 | −1545.6 |
| *Median* | −1564.9 | −1563.9 | −1542.6 | −1543.9 | −1936.2 | −1631.8 | −1531.5 | −1536.5 | −1551.3 | −1558.5 | −1533.3 | −1536.0 |
| $Q_{.25}$ | −1834.2 | −1831.4 | −1826.4 | −1828.7 | −2170.3 | −1898.8 | −1829.9 | −1830.7 | −1830.3 | −1832.5 | −1830.8 | −1829.1 |
| $Q_{.75}$ | −1255.5 | −1254.7 | −1242.9 | −1248.6 | −1614.7 | −1344.6 | −1237.8 | −1237.5 | −1260.6 | −1259.2 | −1236.6 | −1235.3 |
| *Min* | −2238.3 | −2222.0 | −2382.0 | −2232.9 | −3582.0 | −2297.4 | −2292.0 | −2276.9 | −2115.3 | −2281.7 | −4585.5 | −2287.8 |
| *Max* | −920.8 | −919.8 | −901.3 | −924.5 | −1310.2 | −993.5 | −899.5 | −899.5 | −921.2 | −924.0 | −899.7 | −900.3 |
| % ≤ | 87.7 | 83.1 | 53.8 | 100.0 | 100.0 | 100.0 | 70.8 | 100.0 | 73.8 | 90.8 | 70.8 | 100.0 |
| *Panel C: Close-to-close log-returns - In-sample partial likelihood* | | | | | | | | | | | | |
| *Average* | −3297.3 | −3295.5 | −3258.1 | −3262.8 | −3336.1 | −3440.3 | −3257.7 | −3258.1 | −3313.6 | −3271.3 | −3254.9 | −3252.6 |
| *Median* | −3233.4 | −3227.8 | −3201.1 | −3206.0 | −3260.4 | −3392.1 | −3196.7 | −3194.6 | −3245.5 | −3226.6 | −3195.3 | −3190.6 |
| $Q_{.25}$ | −3673.0 | −3673.7 | −3654.4 | −3677.8 | −3710.0 | −3799.3 | −3659.4 | −3663.5 | −3664.2 | −3660.4 | −3659.7 | −3658.8 |
| $Q_{.75}$ | −2938.4 | −2941.1 | −2894.2 | −2900.6 | −2966.4 | −3013.7 | −2885.5 | −2891.4 | −2930.9 | −2925.7 | −2885.3 | −2882.0 |
| *Min* | −4566.8 | −4572.5 | −4555.8 | −4549.6 | −4589.4 | −4630.9 | −4548.6 | −4541.9 | −4550.0 | −4553.4 | −4546.9 | −4540.0 |
| *Max* | −2344.9 | −2341.1 | −2274.7 | −2294.6 | −2368.4 | −2428.4 | −2277.0 | −2278.5 | −2286.8 | −2285.8 | −2271.9 | −2275.7 |
| % ≤ | 93.8 | 93.8 | 26.2 | 100.0 | 96.9 | 98.5 | 44.6 | 100.0 | 87.7 | 81.5 | 67.7 | 100.0 |
| *Panel D: Close-to-close log-returns - Out-of-sample partial likelihood* | | | | | | | | | | | | |
| *Average* | −1737.6 | −1735.8 | −1716.7 | −1717.0 | −2090.8 | −1819.9 | −1712.1 | −1710.3 | −1749.0 | −1746.2 | −1712.2 | −1711.7 |
| *Median* | −1777.3 | −1777.2 | −1724.4 | −1727.5 | −1999.5 | −1884.0 | −1722.4 | −1722.9 | −1818.1 | −1779.8 | −1721.6 | −1724.4 |
| $Q_{.25}$ | −2010.8 | −2008.6 | −1995.3 | −1990.8 | −2374.9 | −2054.0 | −1994.0 | −1990.0 | −2018.8 | −2017.5 | −1992.9 | −1991.3 |
| $Q_{.75}$ | −1410.3 | −1411.1 | −1384.4 | −1382.2 | −1696.5 | −1546.3 | −1378.4 | −1373.7 | −1489.5 | −1398.1 | −1377.1 | −1374.9 |
| *Min* | −2401.4 | −2386.0 | −2429.3 | −2435.8 | −4630.3 | −3049.9 | −2403.6 | −2410.4 | −2499.0 | −2643.2 | −2418.3 | −2411.5 |
| *Max* | −1083.0 | −1082.1 | −1064.5 | −1064.4 | −1313.0 | −1139.2 | −1058.2 | −1058.4 | −1074.3 | −1075.9 | −1058.4 | −1059.8 |
| % ≤ | 93.8 | 93.8 | 43.1 | 100.0 | 100.0 | 98.5 | 63.1 | 100.0 | 61.5 | 93.8 | 53.8 | 100.0 |

Table 5 provides results for three large classes of models including benchmark GARCH models. Reported statistics were computed for equivalent specification of each model, i.e. GARCH(1,1),FIGARCH(1,d,1), RealGARCH(1,1), FloGARCH(1,d,1), etc. The last line of each panel provides the percentage of stocks from the data base for which the likelihood of FloGARCH is higher or equal, be it in-sample or out-of-sample. E.g. in Panel A, the GARCH model: 95.4 means that for 95.4% of the stocks, the FloGARCH has a higher likelihood.

# 5 Forecasting

FloGARCH models allow to construct multi-step ahead forecasts for the latent volatility process of financial securities. The first part of this section introduces the forecasting algorithms. Performances are then compared with competing models. Two Realized GARCH models are included in the set of competing models together with two long-memory benchmarks, namely the HAR-RV model of Corsi (2009) and the ARFIMA specification suggested by Andersen *et al.* (2003). Multiple forecasts comparison is performed on the basis of the Model Confidence Set of Hansen *et al.* (2011) and by comparing the $R^2$ computed from Mincer-Zarnowitz regressions (see e.g. Patton and Sheppard (2009)). Finally, open-to-close returns are used in Section 5 and all models are specified as the usual (1,d,1).[6]

## 5.1 Forecasting with FloGARCH models

**Linear FloGARCH and FloLGARCH**. Denoting by $\check{h}_t$ and $\check{x}_t$ the sequences $h_t$ and $x_t$ or their logarithmic transformation, the equation for the k-steps ahead observation is written as

$$\check{h}_{t+k} = \omega + \sum_{j\geq 1} \lambda_j \check{x}_{t+k-j},$$

$$\check{x}_{t+k} = \xi + \varphi \check{h}_{t+k} + \delta(z_{t+k}) + u_{t+k},$$

where $\delta(z_{t+k}) + u_{t+k}$ is a martingale difference sequence leading to $\check{x}_{t+k|t} = \xi + \varphi \check{h}_{t+k|t}$. Recursive forecasts for the conditional variance of returns can then be extracted from the previous system as

$$\check{h}_{t+k|t} = \left[\omega + \xi \sum_{j=1}^{k-1} \lambda_j\right] + \varphi \sum_{j=1}^{k-1} \lambda_j \check{h}_{t+k-j|t} + \sum_{j\geq k} \lambda_j \check{x}_{t+k-j}.$$

The previous equations provide a general framework for FloGARCH(p,d,q) and FloLGARCH(p,d,q). The coefficients of the infinite filter can be adapted to the model specification. Three remarks are in order. First, the infinite polynomial $\lambda(z)$ has to be truncated in order to compute forecasts and initial values need to be provided for the recursion. Following empirical results from Baillie *et al.* (1996), Bollerslev and Mikkelsen (1996), a truncation level of 1000 is used together with the same initial conditions as in Section 3. Second, the recursive algorithm provides only sufficient tools to forecast $\check{h}_{t+k}$ while the object of interest lies in $h_{t+k}$. This is specific to the FloLGARCH and FloEGARCH models and more details are provided below. Finally, given the arguments against the linear FloGARCH provided in Subsection 4.1.1, we focus on the FloLGARCH.

**FloEGARCH**. We denote $\check{h}_t = \log h_t$ and $\check{x}_t = \log x_t$ and as for the previous section, the k-steps ahead observation is written

$$\check{h}_{t+k} = \omega + \beta(L)^{-1}(1-L)^{-d}\left[\alpha(L)\tau(z_{t+k}) + \gamma(L)u_{t+k}\right],$$

$$\check{x}_{t+k} = \xi + \varphi \check{h}_{t+k} + \delta(z_{t+k}) + u_{t+k}.$$

[6] We use the following notation to denote k-steps ahead forecast: $E[\check{h}_{t+k}|\mathcal{F}_t] = \check{h}_{t+k|t}$.

The innovations $\tau(z_{t+k})$ and $u_{t+k}$ are both martingale difference sequences. Moreover, the GARCH equation can be restated using infinite filters $\breve{h}_{t+k} = \omega + \sum_{j\geq 1} \lambda_j \tau(z_{t+k-j}) + \sum_{i\geq 1} \psi_i u_{t+k-i}$, which allows to compute forecasts as

$$\breve{h}_{t+k|t} = \omega + \sum_{j\geq k} \lambda_j \tau(z_{t+k-j}) + \sum_{i\geq k} \psi_i u_{t+k-i}.$$

Notice again that forecasting $\breve{h}_{t+k}$ is not central to our approach but will be useful to extract information about expected values for $h_{t+k}$.

**Simulation and bootstrap predictions**. Modeling the logarithmic volatilities instead of the volatilities avoids parameter constraints ensuring non-negative conditional variances. However, Jensen's inequality implies that $E[\log h_{t+k}|\mathcal{F}_t] \leq \log E[h_{t+k}|\mathcal{F}_t]$ and prevents direct forecasts for volatilities. Nonetheless, assuming a probability distribution on the residuals, formulas can be derived for some models such as the EGARCH (see Tsay (2005)). Otherwise, numerical methods have to be used to extract multi-step ahead forecasts. We describe two procedures.

First, simulations can be constructed from the model. Based on the normal distribution, the variables can be generated from the system

$$\zeta_{t+k} := \begin{pmatrix} z_{t+k} \\ u_{t+k} \end{pmatrix} \sim N_2 \left( 0, \begin{bmatrix} 1 & 0 \\ 0 & \sigma_u^2 \end{bmatrix} \right), \qquad k = 1, ..., H.$$

From these variables and the estimated model, $\breve{h}_{t+k}$ can be computed at different horizons. If one generates N paths for the log-volatility process, consistent estimates of $h_{t+k}$ can be obtained at each horizon from $\frac{1}{N}\sum_{i=1}^{N} exp(\breve{h}_{t+k})$. In spite of its simplicity, the Gaussian distribution is often a questionable assumption for the joint distribution of residuals and other distributions or procedures may be preferred. Nevertheless, it is noteworthy to mention that the Gaussian distribution may be useful for some asset classes as pointed out by Hansen *et al.* (2014a).

Second, the bootstrap provides a simple distribution-free technique for which the methodology remains essentially similar. $\zeta_{t+k}$ is randomly generated by re-sampling the estimated residuals from the model $(\hat{\zeta}_1, ..., \hat{\zeta}_t)$. Based on evidences from Subsection 4.1.2, we use the bootstrap procedure in this paper.

## 5.2 Forecast evaluation and empirical results

Following studies from Andersen and Bollerslev (1998) and Hansen and Lunde (2005a), our procedure to compare forecasts is based on Mincer-Zarnowitz regressions, the mean-squared error (MSE) and the model confdence set of Hansen *et al.* (2011). Patton (2011) studied forecast evaluation for unobserved variables based on imperfect proxies. He defines a loss function as robust if it yields an equivalent ranking of competing forecasts when evaluated using an unbiased proxy or the true object of interest. We use the MSE loss function that is robust and is provided by the following expression:

$$L(\hat{\sigma}_t^2, h_t) = (h_t - \hat{\sigma}_t^2)^2,$$

where $\hat{\sigma}_t^2$ and $h_t$ denote respectively the proxy of the latent volatility and the forecast for the same period.

The MSE losses are used as inputs for the Model Confidence Set of Hansen *et al.* (2011) (MCS henceforth). The MCS is based on recursive testing and elimination of poor forecasting models. Starting from a set of models $\mathcal{M}_0$ used to compute multi-step ahead forecasts, the MCS tests the null that all the models are indistinguishable in terms of forecasting performance (i.e. this is the *equivalence test* $\delta_{\mathcal{M}}$). If $H_{0,\mathcal{M}}$ is rejected, the MCS removes one forecasting model from the set of models $\mathcal{M}_0$ with an *elimination rule* $\mathfrak{e}_{\mathcal{M}}$. The algorithm proceeds recursively until a non-rejection of $H_{0,\mathcal{M}}$ providing a data-driven optimal set of models $\hat{\mathcal{M}}^{\star}_{1-\alpha}$ that are statistically not distinguishable in terms of forecasting losses. The analysis is performed for all stocks and we report the percentage of time each model was included in the MCS at 5% level. Results of this section are based on 5-min subsampled realized volatilities. Robustness checks by using different proxies and additional results are reported in Appendix 7.3.

Table 6: Forecasting Results.

| *Horizon* | 1 | 2 | 3 | 4 | 5 | 10 | 15 | 20 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| *Panel A: Model Confidence Set results – MSE* | | | | | | | | | |
| $HAR-RV$ | 0.38 | 0.43 | 0.37 | 0.43 | 0.35 | 0.29 | 0.23 | 0.28 | 0.22 |
| $ARFIMA$ | 0.35 | 0.43 | 0.40 | 0.38 | 0.35 | 0.34 | 0.32 | 0.32 | 0.31 |
| $RealLGARCH$ | 0.11 | 0.08 | 0.05 | 0.08 | 0.06 | 0.09 | 0.11 | 0.09 | 0.12 |
| $RealEGARCH$ | 0.20 | 0.12 | 0.14 | 0.12 | 0.12 | 0.25 | 0.22 | 0.20 | 0.25 |
| $FloLGARCH$ | 0.54 | 0.40 | 0.35 | 0.34 | 0.35 | 0.31 | 0.20 | 0.14 | 0.23 |
| $FloEGARCH$ | 0.43 | 0.34 | 0.35 | 0.35 | 0.37 | 0.45 | 0.43 | 0.46 | 0.58 |
| *Panel B: $R^2$ from Mincer-Zarnowitz regressions* | | | | | | | | | |
| $HAR-RV$ | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.46 | 0.40 | 0.38 | 0.32 |
| $ARFIMA$ | 0.24 | 0.36 | 0.36 | 0.37 | 0.37 | 0.37 | 0.32 | 0.29 | 0.25 |
| $RealLGARCH$ | 0.51 | 0.47 | 0.45 | 0.45 | 0.45 | 0.44 | 0.38 | 0.35 | 0.31 |
| $RealEGARCH$ | 0.52 | 0.47 | 0.45 | 0.45 | 0.44 | 0.44 | 0.38 | 0.34 | 0.30 |
| $FloLGARCH$ | 0.55 | 0.53 | 0.51 | 0.50 | 0.51 | 0.48 | 0.43 | 0.38 | 0.37 |
| $FloEGARCH$ | 0.57 | 0.54 | 0.53 | 0.53 | 0.54 | 0.52 | 0.48 | 0.44 | 0.40 |

Panel A summarizes percentage of stocks over the data set for which the model is included in the MCS at 5% at different horizons. Panel B summarizes the $R^2$ from Mincer-Zarnowitz regressions. Results in both panels are based on 5-min subsampled realized volatilities.

Forecasts are computed for 40 periods ahead and the initial sample for the pseudo out-of-sample forecasting exercise contains data from January 2002 to December 2008. A rolling window strategy based on a window size of 1500 observations is used. The main results can be found in Figures 5 and 6 and in Table 6. The Mincer-Zarnowitz regression uses forecasts as regressors and the volatility proxies as dependent variables. The regression provides $R^2$ that allow to gauge the predicting power of the forecast on the proxy. It can be used for multiple models comparison and higher $R^2$ suggest better performances. The Mincer-Zarnowitz regression was computed for all assets of the data set and Figure 5 reports the average $R^2$. The left panel provides results with for the FloLGARCH. It provides for many periods higher average values than the competing models. The HAR model of Corsi (2009) performs almost as well while the ARFIMA provides the least convincing results. On the right panel, performances of

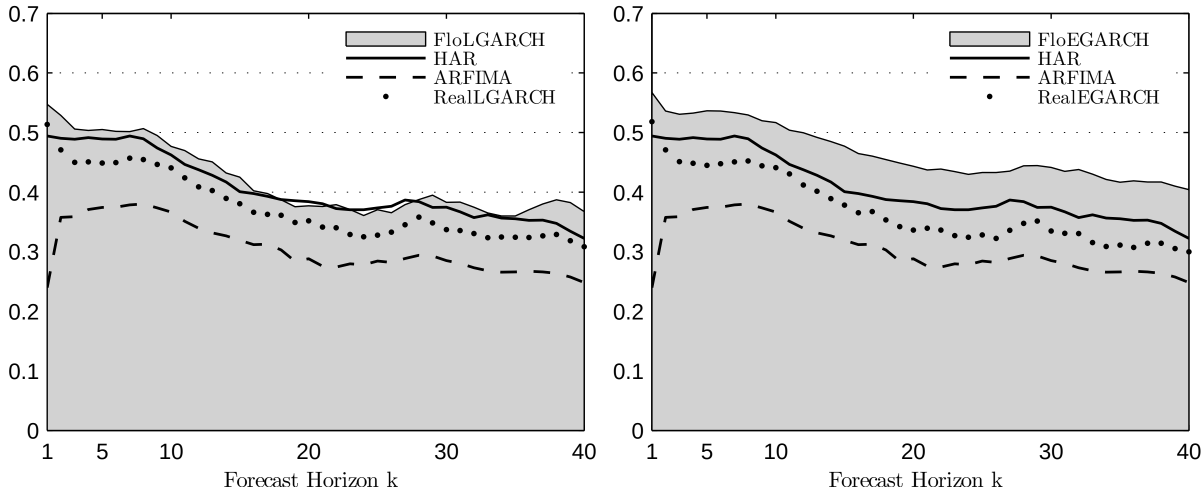


Figure 5: Mincer-Zarnowitz $R^2$ for FloLGARCH(1,d,1) and FloEGARCH(1,d,1).

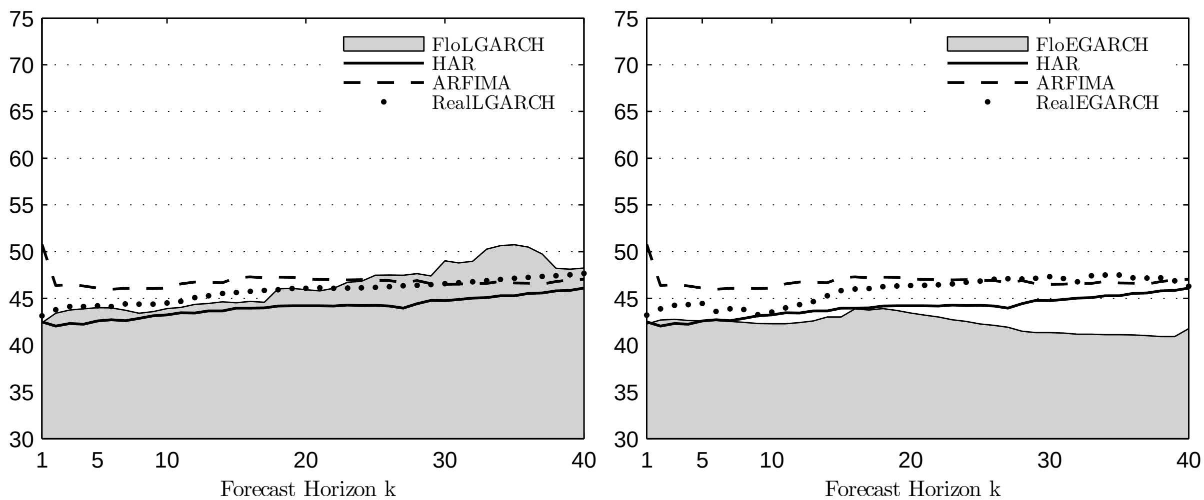


Figure 6: Mean-squared error for FloLGARCH(1,d,1) and FloEGARCH(1,d,1).

the FloEGARCH are reported against competing models and it appears more clearly that the model outperforms all the competing models at all horizons. It can be seen from Table 6 that the FloEGARCH provides uniformly higher $R^2$ than the competing models. These results are robust to the volatility proxy used in the regression and additional figures in Appendix 7.3 illustrate the superior performance of FloGARCH models.

Figure 6 reports the average MSE of the forecasts at the different horizons computed over the data base of stocks. As expected, the values increase with the horizon, i.e. forecasts become less precise for longer horizons. The FloLGARCH is outperformed by the HAR model that has uniformly smaller MSE. On the other hand, the right panel suggests that the FloEGARCH provides better forecasting precision than competing models except for the HAR model and at very short horizons. These two observations are confirmed by results reported in Table 6, which report the percentage of stocks for which the model is included in the MCS. Clearly, the HAR model provides good results for short horizons (i.e. 1 to 5 periods ahead) but the FloEGARCH performs better at the remaining horizons. Conclusions based on the two first parts of the analysis suggest that the FloGARCH models are a serious class of competing models to predict markets volatility.

## 6 Conclusion

This paper introduces a new class of long-memory models for the joint-dynamics of low-frequency returns and realized measures. The class of model is called FloGARCH and includes three different models, the linear FloGARCH, the FloLGARCH and the FloEGARCH. The latter is flexible enough to capture asymmetric shocks between volatility and returns. FloGARCH models can be estimated using Gaussian-QMLE and the estimation procedure is accurate and straightforward to implement. A numerical analysis underlines the reliability of the methodology and the desirable in-sample properties of the estimated parameters. We present empirical evidences about the usefulness of the models and their superior performance. In-sample and out-of sample likelihood measures are used to show the higher ability of FloGARCH to fit historical data. The models are tested with various realized measures and parameters constraints. A pseudo out-of-sample forecasting exercise shows that the FloGARCH models provide more accurate forecasts than benchmark long-memory models and Realized GARCH models.

Finally, we see three sensible directions for future research. First, extending the theoretical properties of Realized ARCH($\infty$) models. These properties include stationary solutions and asymptotic theory for QMLE. Second, our FloGARCH models may be further developed to include several realized measures and a conditional mean similarly to Christensen *et al.* (2010a). The latter development would provide a convenient framework to study the relationships between the historical stock market premium and realized measures. Finally, following recent developments in factor models, FloGARCH models could be extended using a factor structure for residuals similar in spirit to the Realized Beta GARCH model of Hansen *et al.* (2014b). It would then provide a useful tool for the analysis of large dimensional conditional covariance matrices capturing intrinsic long-run relationships among conditional correlations and betas.

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# 7 Appendix: supplementary material

## 7.1 Coefficients of FloGARCH(1,d,1) and FloLGARCH(1,d,1)

The coefficients of the infinite polynomial can be recursively computed. The fractional differencing operator $(1-L)^d$ can be expressed as

$$(1-L)^d = \sum_{k=0}^{\infty} \delta_{d,k} L^k.$$

The coefficients are computed as $\delta_{d,k} = \delta_{d,k-1}\frac{k-1-d}{k}$ where $\delta_{d,0} = 1$. From section 2, the filter of Flo(L)GARCH(p,d,q) is written as $\lambda(L) = 1-\gamma(L)(1-\beta(L))^{-1}(1-L)^d$, which gives for Flo(L)GARCH(1,d,1) $\lambda(L) = 1-(1-\gamma L)(1-\beta L)^{-1}(1-L)^d$. From this expression one has

$$\begin{aligned}\lambda(L) &= 1-(1-\gamma L)\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}\beta^j\delta_{d,k-j}\Big)L^k,\\ &= 1-\sum_{k=0}^{\infty}\psi_k\Big(L^k-\gamma L^{k+1}\Big).\end{aligned}$$

From the previous filter one can observe that $\forall k \geq 1 : \lambda_k = -\psi_k + \gamma\psi_{k-1}$ and $\lambda_0 = -\psi_0 + 1 = 0$, which provides the sufficient recursion to compute the coefficients of $\lambda(L)$. Notice that Caporin (2003) has provided conditions on parameters of FIGARCH ensuring non-negativity of the latent volatility process that are used for the FloGARCH specification.

## 7.2 Coefficients of FloEGARCH(1,d,1)

The FloEGARCH(1,d,1) is expressed as

$$\check{h}_t = \omega + (1-\beta L)^{-1}(1-L)^{-d}\Big[\tau(z_{t-1}) + \gamma u_{t-1}\Big].$$

The infinite polynomial is denoted by $(1-\beta L)^{-1}(1-L)^{-d}$ and coefficients computation follows along the same line as in the previous Appendix. Denoting $c = -d$ allows to write $(1-L)^c = \sum_{k=0}^{\infty}\delta_{c,k}L^k$ where $\delta_{c,k} = \delta_{c,k-1}\frac{k-1-c}{k} = \delta_{-d,k-1}\frac{k-1+d}{k}$. This simple trick leads to

$$(1-\beta L)^{-1}(1-L)^{-d} = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}\beta^j\delta_{-d,k-j}\Big)L^k.$$

The filter can be computed for any specification following the same strategy or from Proposition 2 of Lopes and Prass (2014).

## 7.3 Further estimation results

This section gathers supplementary estimation results that were not included in the main text to save some space. In order of appearance:

- Table 7: Estimation results for FloLGARCH(1,d,1) based on $r_t^{oc}$.
- Table 8: Estimation results for FloEGARCH(1,d,1) based on $r_t^{oc}$.
- Table 9: Comparison table for different realized measures based on $r_t^{cc}$.
- Table 10: Comparison table for constraint versions of the models based on $r_t^{cc}$.
- Figures 7 to 9: Robustness checks for Mincer-Zarnowitz regressions.

Table 7: Estimation results - FloLGARCH(1,d,1) - $r_t^{oc}$.

| | $\mu$ | $\omega$ | $\gamma$ | $d$ | $\beta$ | $\xi$ | $\varphi$ | $\delta_1$ | $\delta_2$ | $l(r,x)$ | $l(r)$ | $\hat{\sigma}_u^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AA | −0.13 | 0.04 | 0.15 | 0.64 | 0.32 | −0.01 | 0.99 | −0.04 | 0.08 | −6161.7 | −5071.6 | 0.14 |
| ABT | 0.05 | −0.07 | 0.29 | 0.60 | 0.45 | 0.06 | 1.00 | −0.02 | 0.06 | −5097.6 | −3806.8 | 0.17 |
| AES | −0.00 | 0.20 | 0.39 | 0.83 | 0.71 | −0.15 | 0.97 | −0.03 | 0.07 | −7184.3 | −5520.6 | 0.23 |
| AIG | −0.09 | 0.23 | 0.68 | 0.78 | 0.84 | −0.20 | 0.98 | −0.02 | 0.05 | −6834.6 | −5199.3 | 0.22 |
| AKS | −0.17 | 0.38 | 0.79 | 0.28 | 0.73 | −0.55 | 1.15 | −0.04 | 0.09 | −8295.2 | −6451.8 | 0.26 |
| AMD | −0.13 | 0.20 | 0.48 | 0.57 | 0.61 | −0.24 | 1.03 | −0.02 | 0.09 | −7305.2 | −6062.4 | 0.16 |
| AXP | 0.04 | 0.10 | −0.20 | 0.63 | −0.07 | −0.10 | 1.01 | −0.03 | 0.08 | −5717.7 | −4530.9 | 0.15 |
| BA | 0.00 | 0.01 | −0.44 | 0.51 | −0.37 | −0.03 | 1.03 | −0.03 | 0.07 | −5449.1 | −4363.0 | 0.14 |
| BAC | 0.01 | 0.12 | 0.52 | 0.79 | 0.70 | −0.10 | 0.98 | −0.03 | 0.09 | −5760.6 | −4555.1 | 0.16 |
| BMY | −0.01 | −0.13 | 0.38 | 0.68 | 0.63 | 0.13 | 0.99 | −0.04 | 0.09 | −5396.1 | −3948.3 | 0.19 |
| BSX | −0.07 | 0.19 | 0.50 | 0.44 | 0.60 | −0.23 | 1.05 | −0.02 | 0.12 | −6846.2 | −5082.0 | 0.25 |
| C | −0.07 | 0.06 | 0.17 | 0.66 | 0.25 | −0.04 | 0.98 | −0.03 | 0.10 | −6060.1 | −4786.3 | 0.17 |
| CAG | 0.01 | −0.08 | 0.37 | 0.56 | 0.55 | 0.08 | 1.01 | −0.03 | 0.08 | −5073.1 | −3544.3 | 0.20 |
| CAT | 0.01 | 0.07 | 0.40 | 0.78 | 0.65 | −0.05 | 0.98 | −0.04 | 0.07 | −5628.9 | −4607.1 | 0.13 |
| CHK | −0.06 | 0.12 | 0.36 | 0.75 | 0.66 | −0.07 | 0.97 | −0.05 | 0.08 | −6742.1 | −5296.7 | 0.19 |
| CLF | 0.10 | 0.43 | 0.09 | 0.47 | 0.19 | −0.48 | 1.05 | −0.02 | 0.10 | −7885.7 | −6060.5 | 0.26 |
| COH | 0.03 | 0.19 | 0.37 | 0.68 | 0.63 | −0.18 | 0.99 | −0.01 | 0.11 | −6678.8 | −5233.1 | 0.19 |
| CSX | 0.06 | 0.11 | 0.31 | 0.64 | 0.52 | −0.12 | 1.00 | −0.02 | 0.07 | −5950.9 | −4773.3 | 0.15 |
| D | 0.03 | −0.06 | −0.21 | 0.61 | −0.06 | 0.06 | 1.00 | −0.02 | 0.07 | −4838.2 | −3537.4 | 0.17 |
| DD | 0.03 | −0.09 | 0.25 | 0.71 | 0.45 | 0.10 | 0.98 | −0.05 | 0.06 | −5269.5 | −4133.9 | 0.15 |
| DIS | 0.08 | −0.06 | 0.39 | 0.72 | 0.61 | 0.07 | 0.99 | −0.05 | 0.08 | −5373.2 | −4229.4 | 0.15 |
| DNR | 0.01 | 0.17 | 0.17 | 0.71 | 0.55 | −0.16 | 0.99 | −0.03 | 0.10 | −7407.6 | −5556.5 | 0.27 |
| DOW | 0.02 | 0.01 | 0.27 | 0.76 | 0.54 | 0.01 | 0.97 | −0.05 | 0.07 | −5916.9 | −4648.6 | 0.16 |
| EMC | 0.10 | −0.02 | 0.57 | 0.61 | 0.69 | 0.02 | 1.00 | −0.02 | 0.09 | −6104.2 | −4979.9 | 0.15 |
| EXC | −0.01 | −0.03 | 0.15 | 0.68 | 0.41 | 0.02 | 1.00 | −0.03 | 0.08 | −5266.4 | −3984.3 | 0.17 |
| F | −0.16 | 0.13 | 0.51 | 0.68 | 0.71 | −0.12 | 0.99 | −0.03 | 0.09 | −6975.7 | −5302.1 | 0.23 |
| FCX | −0.05 | 0.21 | 0.25 | 0.54 | 0.32 | −0.24 | 1.03 | −0.05 | 0.08 | −6762.3 | −5615.7 | 0.15 |
| GE | −0.01 | −0.04 | 0.28 | 0.71 | 0.50 | 0.05 | 0.99 | −0.02 | 0.07 | −5276.3 | −4114.0 | 0.15 |
| GIS | 0.02 | −0.09 | 0.51 | 0.70 | 0.75 | 0.08 | 0.99 | 0.01 | 0.06 | −4568.0 | −3227.6 | 0.17 |
| GLW | −0.06 | 0.10 | 0.45 | 0.69 | 0.64 | −0.07 | 0.98 | −0.04 | 0.09 | −6734.3 | −5409.7 | 0.17 |
| HAL | 0.00 | −0.01 | 0.28 | 0.78 | 0.58 | 0.05 | 0.97 | −0.04 | 0.08 | −6404.9 | −5303.1 | 0.14 |
| HD | 0.02 | −0.01 | 0.38 | 0.73 | 0.63 | 0.02 | 0.99 | −0.03 | 0.07 | −5443.6 | −4368.2 | 0.14 |
| IBM | 0.13 | −0.11 | 0.23 | 0.74 | 0.47 | 0.10 | 0.98 | −0.04 | 0.06 | −4638.2 | −3607.9 | 0.14 |
| INTC | −0.01 | −0.01 | 0.46 | 0.69 | 0.62 | 0.02 | 0.99 | −0.02 | 0.07 | −5543.9 | −4652.9 | 0.12 |
| IRM | 0.03 | 0.31 | 0.32 | 0.36 | 0.30 | −0.36 | 1.07 | −0.03 | 0.06 | −6552.5 | −4456.3 | 0.32 |
| JCP | 0.02 | 0.18 | 0.46 | 0.66 | 0.67 | −0.18 | 1.00 | −0.02 | 0.06 | −6680.4 | −5340.8 | 0.17 |
| JNJ | 0.02 | −0.21 | 0.32 | 0.82 | 0.65 | 0.19 | 0.97 | 0.00 | 0.06 | −4246.8 | −3017.3 | 0.16 |
| JPM | 0.01 | 0.07 | 0.42 | 0.78 | 0.63 | −0.06 | 0.99 | −0.03 | 0.08 | −5787.2 | −4706.4 | 0.14 |
| KEY | −0.01 | 0.02 | 0.29 | 0.71 | 0.52 | −0.00 | 0.99 | −0.02 | 0.09 | −6204.9 | −4772.6 | 0.19 |
| KO | 0.05 | −0.15 | 0.33 | 0.61 | 0.47 | 0.14 | 1.00 | −0.02 | 0.07 | −4424.2 | −3259.2 | 0.15 |
| MCD | 0.06 | −0.06 | 0.34 | 0.70 | 0.64 | 0.05 | 0.99 | −0.03 | 0.09 | −5093.0 | −3795.0 | 0.17 |
| MDT | 0.03 | −0.06 | 0.25 | 0.65 | 0.47 | 0.06 | 0.98 | −0.03 | 0.07 | −5087.5 | −3853.8 | 0.16 |
| MMM | 0.01 | −0.09 | 0.33 | 0.72 | 0.55 | 0.10 | 0.98 | −0.04 | 0.06 | −4885.6 | −3678.0 | 0.16 |
| MO | 0.02 | −0.02 | 0.33 | 0.90 | 0.77 | 0.03 | 0.95 | −0.01 | 0.06 | −5217.4 | −3666.1 | 0.21 |
| MRK | −0.01 | 0.01 | 0.41 | 0.65 | 0.62 | −0.01 | 1.00 | −0.03 | 0.05 | −5574.1 | −4130.3 | 0.19 |
| MSFT | 0.04 | −0.09 | 0.31 | 0.73 | 0.55 | 0.10 | 0.97 | −0.04 | 0.07 | −5014.9 | −4025.4 | 0.13 |
| NBR | −0.09 | 0.19 | 0.33 | 0.64 | 0.54 | −0.20 | 1.01 | −0.03 | 0.07 | −6652.5 | −5523.6 | 0.15 |
| NEM | −0.09 | 0.09 | 0.40 | 0.66 | 0.57 | −0.08 | 1.00 | −0.03 | 0.07 | −5774.2 | −5043.8 | 0.11 |
| ORCL | 0.07 | −0.05 | 0.57 | 0.66 | 0.72 | 0.06 | 0.99 | −0.02 | 0.09 | −5802.8 | −4697.4 | 0.14 |
| PFE | −0.03 | −0.09 | 0.44 | 0.67 | 0.64 | 0.09 | 0.99 | −0.04 | 0.08 | −5131.5 | −3993.4 | 0.15 |
| PG | 0.07 | −0.21 | 0.36 | 0.65 | 0.53 | 0.20 | 0.98 | −0.04 | 0.06 | −4368.9 | −3103.7 | 0.16 |
| S | −0.01 | 0.18 | 0.48 | 0.64 | 0.68 | −0.17 | 1.01 | −0.01 | 0.09 | −7155.2 | −5544.8 | 0.22 |
| SLB | 0.03 | −0.00 | 0.35 | 0.72 | 0.58 | 0.01 | 1.00 | −0.04 | 0.06 | −5868.1 | −4998.8 | 0.12 |
| SPY | 0.00 | 0.16 | 0.29 | 0.75 | 0.47 | −0.17 | 0.98 | −0.08 | 0.06 | −4131.8 | −3094.3 | 0.14 |
| T | −0.02 | −0.03 | 0.45 | 0.77 | 0.70 | 0.04 | 0.97 | −0.03 | 0.08 | −5388.0 | −4067.3 | 0.17 |
| TJX | 0.08 | −0.01 | −0.09 | 0.53 | 0.04 | 0.01 | 1.02 | −0.00 | 0.07 | −5815.6 | −4453.3 | 0.18 |
| USB | 0.05 | −0.06 | 0.28 | 0.72 | 0.47 | 0.07 | 0.99 | −0.03 | 0.08 | −5661.7 | −4290.0 | 0.18 |
| UTX | 0.00 | −0.06 | 0.27 | 0.67 | 0.47 | 0.06 | 0.98 | −0.03 | 0.07 | −5087.9 | −3903.2 | 0.15 |
| VLO | −0.02 | 0.20 | 0.32 | 0.70 | 0.56 | −0.20 | 1.00 | −0.02 | 0.09 | −6484.8 | −5284.0 | 0.16 |
| VZ | −0.02 | −0.13 | 0.27 | 0.68 | 0.50 | 0.13 | 0.99 | −0.03 | 0.07 | −5111.1 | −3853.8 | 0.16 |
| WFC | −0.00 | −0.03 | 0.37 | 0.77 | 0.61 | 0.04 | 0.99 | −0.03 | 0.08 | −5441.7 | −4326.0 | 0.15 |
| WMT | 0.00 | −0.18 | 0.28 | 0.67 | 0.50 | 0.18 | 1.00 | −0.01 | 0.07 | −4500.2 | −3494.0 | 0.13 |
| WY | −0.02 | 0.13 | 0.33 | 0.64 | 0.52 | −0.13 | 1.00 | −0.03 | 0.08 | −5867.3 | −4748.9 | 0.15 |
| XOM | 0.07 | −0.09 | 0.28 | 0.80 | 0.55 | 0.09 | 0.98 | −0.07 | 0.06 | −4823.4 | −3837.4 | 0.13 |
| XRX | 0.11 | 0.05 | 0.30 | 0.63 | 0.51 | −0.03 | 0.99 | −0.03 | 0.08 | −6487.4 | −4893.1 | 0.22 |

Table 8: Estimation results - FloEGARCH(1,d,1) - $r_t^{oc}$.

| | $\mu$ | $\omega$ | $d$ | $\beta$ | $\tau_1$ | $\tau_2$ | $\gamma$ | $\xi$ | $\varphi$ | $\delta_1$ | $\delta_2$ | $l(r,x)$ | $l(r)$ | $\hat{\sigma}_u^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AA | −0.14 | 1.26 | 0.66 | 0.18 | −0.07 | 0.06 | 0.40 | −0.07 | 1.03 | −0.05 | 0.07 | −6123.8 | −5066.3 | 0.14 |
| ABT | 0.04 | 0.43 | 0.68 | 0.09 | −0.05 | 0.04 | 0.41 | 0.07 | 0.99 | −0.02 | 0.06 | −5079.6 | −3803.8 | 0.17 |
| AES | −0.02 | 2.51 | 0.71 | 0.02 | −0.08 | 0.06 | 0.45 | −0.04 | 0.91 | −0.03 | 0.08 | −7141.3 | −5502.9 | 0.22 |
| AIG | −0.07 | 0.30 | 0.63 | 0.10 | −0.07 | 0.05 | 0.59 | −0.07 | 0.90 | −0.01 | 0.05 | −6768.5 | −5158.9 | 0.22 |
| AKS | −0.13 | 1.89 | 0.66 | −0.01 | −0.05 | 0.05 | 0.26 | −0.93 | 1.31 | −0.04 | 0.09 | −8241.5 | −6447.6 | 0.25 |
| AMD | −0.15 | 2.48 | 0.67 | 0.01 | −0.03 | 0.06 | 0.32 | −0.68 | 1.24 | −0.02 | 0.10 | −7269.3 | −6054.5 | 0.16 |
| AXP | 0.03 | 1.34 | 0.67 | 0.21 | −0.08 | 0.06 | 0.43 | −0.14 | 1.04 | −0.03 | 0.08 | −5665.3 | −4531.4 | 0.15 |
| BA | −0.01 | 0.96 | 0.67 | 0.14 | −0.06 | 0.05 | 0.33 | −0.11 | 1.14 | −0.03 | 0.07 | −5404.0 | −4356.0 | 0.14 |
| BAC | 0.01 | 1.17 | 0.68 | 0.09 | −0.08 | 0.09 | 0.54 | −0.07 | 0.95 | −0.04 | 0.08 | −5692.7 | −4538.2 | 0.15 |
| BMY | −0.01 | 0.87 | 0.70 | 0.01 | −0.04 | 0.05 | 0.39 | 0.13 | 1.01 | −0.04 | 0.09 | −5393.6 | −3945.1 | 0.19 |
| BSX | −0.07 | 1.41 | 0.63 | 0.05 | −0.03 | 0.05 | 0.29 | −0.39 | 1.18 | −0.02 | 0.12 | −6831.1 | −5078.8 | 0.24 |
| C | −0.06 | 1.13 | 0.66 | 0.17 | −0.07 | 0.09 | 0.52 | −0.02 | 0.97 | −0.03 | 0.09 | −6014.7 | −4773.1 | 0.16 |
| CAG | 0.00 | 0.11 | 0.69 | −0.00 | −0.04 | 0.04 | 0.34 | 0.08 | 1.04 | −0.03 | 0.08 | −5058.7 | −3537.3 | 0.20 |
| CAT | 0.01 | 0.95 | 0.70 | 0.05 | −0.06 | 0.06 | 0.42 | −0.15 | 1.10 | −0.04 | 0.07 | −5598.8 | −4602.1 | 0.13 |
| CHK | −0.07 | 1.51 | 0.67 | 0.06 | −0.06 | 0.08 | 0.36 | −0.09 | 0.99 | −0.04 | 0.08 | −6685.3 | −5283.5 | 0.18 |
| CLF | 0.11 | 1.19 | 0.66 | 0.11 | −0.07 | 0.07 | 0.26 | −0.64 | 1.16 | −0.02 | 0.10 | −7785.7 | −6010.7 | 0.25 |
| COH | 0.03 | 1.54 | 0.68 | 0.01 | −0.07 | 0.07 | 0.34 | −0.27 | 1.05 | −0.01 | 0.11 | −6636.7 | −5230.2 | 0.18 |
| CSX | 0.06 | 1.19 | 0.71 | 0.08 | −0.05 | 0.04 | 0.34 | −0.28 | 1.16 | −0.02 | 0.07 | −5916.5 | −4766.0 | 0.15 |
| D | 0.03 | 0.06 | 0.67 | 0.24 | −0.04 | 0.05 | 0.39 | 0.06 | 1.03 | −0.02 | 0.07 | −4813.1 | −3527.9 | 0.17 |
| DD | 0.03 | 0.56 | 0.66 | 0.17 | −0.07 | 0.05 | 0.46 | 0.10 | 0.98 | −0.05 | 0.06 | −5240.5 | −4134.2 | 0.14 |
| DIS | 0.06 | 1.44 | 0.69 | 0.11 | −0.08 | 0.05 | 0.42 | 0.05 | 1.02 | −0.05 | 0.08 | −5342.0 | −4222.8 | 0.15 |
| DNR | 0.01 | 1.37 | 0.70 | 0.14 | −0.04 | 0.05 | 0.25 | −0.41 | 1.15 | −0.03 | 0.09 | −7375.8 | −5542.5 | 0.26 |
| DOW | 0.01 | 1.31 | 0.67 | 0.15 | −0.09 | 0.06 | 0.44 | 0.05 | 0.93 | −0.05 | 0.07 | −5882.4 | −4643.9 | 0.16 |
| EMC | 0.08 | 1.91 | 0.66 | 0.13 | −0.04 | 0.05 | 0.46 | 0.04 | 0.98 | −0.02 | 0.09 | −6096.4 | −4978.7 | 0.15 |
| EXC | −0.01 | 0.47 | 0.71 | 0.15 | −0.04 | 0.06 | 0.37 | 0.01 | 1.05 | −0.03 | 0.08 | −5241.4 | −3975.5 | 0.16 |
| F | −0.17 | 1.47 | 0.65 | −0.09 | −0.05 | 0.08 | 0.40 | −0.23 | 1.07 | −0.03 | 0.09 | −6930.9 | −5301.2 | 0.22 |
| FCX | −0.06 | 1.73 | 0.66 | 0.12 | −0.07 | 0.06 | 0.33 | −0.60 | 1.24 | −0.06 | 0.08 | −6682.4 | −5600.6 | 0.14 |
| GE | −0.02 | 0.84 | 0.70 | 0.11 | −0.05 | 0.06 | 0.40 | 0.01 | 1.04 | −0.02 | 0.08 | −5235.6 | −4116.2 | 0.15 |
| GIS | 0.02 | 0.07 | 0.67 | 0.00 | −0.05 | 0.04 | 0.42 | 0.09 | 1.02 | 0.00 | 0.07 | −4544.1 | −3221.0 | 0.17 |
| GLW | −0.08 | 2.37 | 0.69 | 0.05 | −0.05 | 0.07 | 0.45 | −0.08 | 0.98 | −0.04 | 0.09 | −6727.0 | −5411.1 | 0.17 |
| HAL | −0.02 | 2.21 | 0.70 | 0.04 | −0.07 | 0.06 | 0.38 | −0.10 | 1.06 | −0.04 | 0.08 | −6368.3 | −5300.4 | 0.14 |
| HD | 0.01 | 0.80 | 0.70 | 0.03 | −0.06 | 0.05 | 0.42 | −0.02 | 1.04 | −0.03 | 0.07 | −5418.7 | −4367.4 | 0.14 |
| IBM | 0.12 | 0.27 | 0.70 | 0.13 | −0.08 | 0.05 | 0.45 | 0.10 | 1.00 | −0.04 | 0.06 | −4601.4 | −3604.9 | 0.13 |
| INTC | −0.03 | 1.76 | 0.68 | 0.08 | −0.06 | 0.06 | 0.48 | 0.02 | 0.98 | −0.02 | 0.07 | −5510.2 | −4651.0 | 0.12 |
| IRM | 0.04 | 0.61 | 0.60 | 0.08 | −0.03 | 0.03 | 0.30 | −0.50 | 1.25 | −0.02 | 0.07 | −6524.3 | −4459.4 | 0.31 |
| JCP | 0.02 | 1.58 | 0.67 | 0.05 | −0.06 | 0.03 | 0.37 | −0.38 | 1.14 | −0.02 | 0.06 | −6650.1 | −5335.0 | 0.17 |
| JNJ | 0.02 | −0.04 | 0.71 | 0.04 | −0.06 | 0.04 | 0.46 | 0.20 | 1.00 | −0.00 | 0.06 | −4224.4 | −3016.7 | 0.16 |
| JPM | 0.00 | 1.55 | 0.73 | 0.03 | −0.07 | 0.07 | 0.49 | −0.07 | 1.00 | −0.04 | 0.08 | −5741.0 | −4699.2 | 0.14 |
| KEY | −0.00 | 0.62 | 0.71 | 0.09 | −0.08 | 0.07 | 0.43 | 0.01 | 0.98 | −0.03 | 0.09 | −6157.2 | −4765.0 | 0.18 |
| KO | 0.05 | 0.08 | 0.68 | 0.12 | −0.05 | 0.05 | 0.44 | 0.14 | 1.01 | −0.02 | 0.07 | −4405.3 | −3255.6 | 0.15 |
| MCD | 0.05 | 0.63 | 0.72 | 0.03 | −0.05 | 0.04 | 0.38 | 0.05 | 1.00 | −0.03 | 0.09 | −5086.3 | −3791.4 | 0.17 |
| MDT | 0.03 | 0.35 | 0.69 | 0.10 | −0.05 | 0.04 | 0.40 | 0.06 | 1.01 | −0.03 | 0.07 | −5078.9 | −3849.5 | 0.16 |
| MMM | 0.01 | 0.30 | 0.68 | 0.11 | −0.07 | 0.03 | 0.43 | 0.08 | 1.05 | −0.04 | 0.06 | −4866.0 | −3678.7 | 0.15 |
| MO | 0.02 | 0.17 | 0.68 | 0.09 | −0.06 | 0.03 | 0.51 | 0.06 | 0.80 | −0.02 | 0.06 | −5192.2 | −3663.9 | 0.20 |
| MRK | −0.00 | 0.31 | 0.71 | −0.02 | −0.05 | 0.02 | 0.39 | −0.08 | 1.14 | −0.03 | 0.05 | −5559.8 | −4123.0 | 0.19 |
| MSFT | 0.03 | 1.05 | 0.67 | 0.10 | −0.06 | 0.06 | 0.48 | 0.12 | 0.92 | −0.04 | 0.07 | −5000.1 | −4023.8 | 0.13 |
| NBR | −0.10 | 1.79 | 0.70 | 0.05 | −0.06 | 0.05 | 0.33 | −0.44 | 1.17 | −0.03 | 0.07 | −6607.0 | −5502.3 | 0.14 |
| NEM | −0.09 | 0.95 | 0.68 | 0.10 | −0.03 | 0.05 | 0.41 | −0.21 | 1.11 | −0.03 | 0.07 | −5760.1 | −5040.9 | 0.11 |
| ORCL | 0.04 | 1.94 | 0.67 | 0.04 | −0.06 | 0.08 | 0.43 | 0.01 | 1.02 | −0.02 | 0.10 | −5766.8 | −4699.2 | 0.14 |
| PFE | −0.04 | 0.53 | 0.68 | 0.08 | −0.04 | 0.05 | 0.41 | 0.06 | 1.07 | −0.04 | 0.08 | −5121.4 | −3991.3 | 0.15 |
| PG | 0.07 | −0.14 | 0.68 | 0.10 | −0.06 | 0.04 | 0.44 | 0.22 | 1.04 | −0.04 | 0.06 | −4356.7 | −3101.2 | 0.16 |
| S | 0.00 | 1.15 | 0.68 | 0.05 | −0.05 | 0.06 | 0.37 | −0.24 | 1.05 | −0.01 | 0.09 | −7125.8 | −5540.5 | 0.21 |
| SLB | 0.02 | 1.40 | 0.69 | 0.03 | −0.06 | 0.05 | 0.38 | −0.24 | 1.20 | −0.04 | 0.06 | −5810.7 | −4989.6 | 0.11 |
| SPY | −0.00 | −0.25 | 0.68 | 0.15 | −0.11 | 0.06 | 0.42 | −0.16 | 1.05 | −0.09 | 0.06 | −4049.4 | −3102.2 | 0.13 |
| T | −0.02 | 1.09 | 0.70 | 0.05 | −0.06 | 0.07 | 0.51 | 0.09 | 0.89 | −0.03 | 0.08 | −5360.3 | −4054.2 | 0.17 |
| TJX | 0.07 | 0.79 | 0.65 | 0.21 | −0.05 | 0.03 | 0.38 | −0.03 | 1.06 | −0.00 | 0.07 | −5792.7 | −4450.2 | 0.18 |
| USB | 0.04 | 0.82 | 0.70 | 0.13 | −0.08 | 0.08 | 0.47 | 0.08 | 0.97 | −0.03 | 0.08 | −5610.9 | −4285.6 | 0.17 |
| UTX | −0.00 | 0.70 | 0.69 | 0.16 | −0.08 | 0.05 | 0.39 | 0.05 | 1.01 | −0.03 | 0.07 | −5032.5 | −3900.2 | 0.15 |
| VLO | −0.02 | 0.91 | 0.69 | 0.03 | −0.04 | 0.07 | 0.34 | −0.47 | 1.20 | −0.02 | 0.09 | −6428.9 | −5266.2 | 0.15 |
| VZ | −0.02 | 0.52 | 0.70 | 0.12 | −0.06 | 0.05 | 0.41 | 0.12 | 1.01 | −0.03 | 0.07 | −5087.4 | −3855.1 | 0.16 |
| WFC | −0.01 | 0.31 | 0.72 | 0.08 | −0.08 | 0.06 | 0.50 | 0.06 | 0.96 | −0.03 | 0.07 | −5395.0 | −4317.7 | 0.14 |
| WMT | −0.01 | 0.47 | 0.69 | 0.10 | −0.04 | 0.05 | 0.39 | 0.17 | 1.05 | −0.01 | 0.07 | −4480.4 | −3491.9 | 0.13 |
| WY | −0.02 | 1.01 | 0.69 | 0.08 | −0.04 | 0.05 | 0.40 | −0.20 | 1.07 | −0.03 | 0.08 | −5852.7 | −4747.1 | 0.14 |
| XOM | 0.06 | 0.57 | 0.68 | 0.14 | −0.08 | 0.06 | 0.43 | 0.06 | 1.08 | −0.08 | 0.06 | −4792.4 | −3833.3 | 0.13 |
| XRX | 0.09 | 1.80 | 0.64 | 0.18 | −0.07 | 0.04 | 0.38 | −0.04 | 0.99 | −0.03 | 0.08 | −6468.1 | −4891.6 | 0.21 |

Table 9: Comparison Table for different realized measures - $r_t^{cc}$.

| | Flo-Lin | | | | Flo-Log | | | | Flo-Exp | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ | d | $\varphi$ | $l^{IS}(r)$ | $l^{OS}(r)$ |
| $RV1m$ | 0.63 | 0.96 | −8076.97 | −3894.09 | 0.69 | 0.99 | −3918.85 | −2076.86 | 0.71 | 1.08 | −3899.54 | −2071.66 |
| $RV5m$ | 0.58 | 0.96 | −7956.43 | −3815.78 | 0.67 | 0.99 | −4257.84 | −2268.44 | 0.70 | 1.06 | −4237.56 | −2276.54 |
| $RV15m$ | 0.55 | 0.95 | −8110.97 | −4124.61 | 0.61 | 1.00 | −4613.08 | −2471.15 | 0.68 | 1.04 | −4584.08 | −2466.64 |
| $ssRV1m$ | 0.63 | 0.96 | −8076.97 | −3894.09 | 0.69 | 0.99 | −3918.85 | −2076.86 | 0.71 | 1.08 | −3899.54 | −2071.66 |
| $ssRV5m$ | 0.58 | 0.95 | −7918.38 | −3957.74 | 0.67 | 0.99 | −4177.07 | −2234.27 | 0.70 | 1.06 | −4148.82 | −2221.36 |
| $ssRV15m$ | 0.61 | 0.94 | −7755.44 | −3830.78 | 0.62 | 1.00 | −4450.13 | −2373.53 | 0.70 | 1.04 | −4417.40 | −2369.55 |
| $BPV1m$ | 0.66 | 0.95 | −8010.24 | −3887.57 | 0.66 | 1.00 | −4035.23 | −2141.76 | 0.71 | 1.09 | −4003.56 | −2146.04 |
| $BPV5m$ | 0.59 | 0.96 | −7886.54 | −3827.18 | 0.65 | 1.00 | −4322.75 | −2308.25 | 0.70 | 1.08 | −4294.57 | −2309.49 |
| $BPV15m$ | 0.60 | 0.94 | −7699.30 | −4118.16 | 0.63 | 0.99 | −4684.98 | −2489.94 | 0.69 | 1.04 | −4652.20 | −2491.14 |
| $MinRV1m$ | 0.65 | 0.94 | −8045.40 | −3909.97 | 0.69 | 0.99 | −4088.77 | −2131.42 | 0.71 | 1.08 | −4056.35 | −2130.82 |
| $MinRV5m$ | 0.57 | 0.96 | −7939.42 | −3820.81 | 0.64 | 1.00 | −4412.26 | −2372.61 | 0.70 | 1.07 | −4381.90 | −2372.96 |
| $MinRV15m$ | 0.59 | 0.94 | −7932.17 | −4180.67 | 0.62 | 0.99 | −4833.27 | −2584.13 | 0.70 | 1.04 | −4803.85 | −2576.36 |
| $ssMinRV1m$ | 0.66 | 0.94 | −8053.66 | −3820.51 | 0.68 | 0.99 | −4009.76 | −2085.31 | 0.71 | 1.08 | −3976.69 | −2084.46 |
| $ssMinRV5m$ | 0.62 | 0.94 | −7769.90 | −3845.18 | 0.67 | 0.99 | −4216.65 | −2266.56 | 0.70 | 1.07 | −4198.74 | −2260.99 |
| $ssMinRV15m$ | 0.67 | 0.92 | −7715.90 | −3893.27 | 0.63 | 1.00 | −4519.56 | −2417.06 | 0.70 | 1.04 | −4480.27 | −2409.10 |
| $prgMinRV$ | 0.59 | 0.96 | −9513.72 | −4551.96 | 0.53 | 1.03 | −4221.17 | −2285.26 | 0.70 | 1.26 | −4198.89 | −2288.31 |
| $MedRV1m$ | 0.66 | 0.95 | −8070.07 | −3841.67 | 0.69 | 0.99 | −4032.87 | −2109.00 | 0.71 | 1.08 | −4019.38 | −2104.88 |
| $MedRV5m$ | 0.59 | 0.96 | −7835.44 | −3818.92 | 0.65 | 1.00 | −4327.75 | −2321.13 | 0.70 | 1.07 | −4296.58 | −2318.38 |
| $MedRV15m$ | 0.61 | 0.94 | −7780.45 | −4075.78 | 0.63 | 0.99 | −4728.01 | −2514.89 | 0.69 | 1.04 | −4694.42 | −2510.91 |
| $ssMedRV1m$ | 0.66 | 0.94 | −8026.32 | −3804.54 | 0.69 | 0.99 | −3975.35 | −2082.35 | 0.71 | 1.10 | −3961.33 | −2067.62 |
| $ssMedRV5m$ | 0.64 | 0.94 | −7747.78 | −3869.62 | 0.68 | 0.99 | −4196.97 | −2250.72 | 0.70 | 1.07 | −4178.28 | −2239.52 |
| $ssMedRV15m$ | 0.65 | 0.92 | −7614.96 | −3815.61 | 0.64 | 1.00 | −4500.85 | −2401.81 | 0.69 | 1.05 | −4460.93 | −2396.05 |
| $prgMedRV$ | 0.56 | 0.96 | −9520.31 | −4556.53 | 0.54 | 1.02 | −4186.22 | −2248.56 | 0.69 | 1.18 | −4145.07 | −2247.44 |
| $RK$ | 0.64 | 0.95 | −7947.46 | −3796.62 | 0.68 | 1.00 | −4044.32 | −2136.27 | 0.70 | 1.08 | −4003.53 | −2134.65 |
| $TTS$ | 0.69 | 0.95 | −7781.17 | −3848.61 | 0.63 | 1.00 | −4040.27 | −2232.58 | 0.70 | 1.11 | −4024.34 | −2219.94 |
| $MSC$ | 0.69 | 0.95 | −7838.97 | −3821.58 | 0.65 | 1.00 | −3964.54 | −2096.21 | 0.70 | 1.11 | −3942.58 | −2094.58 |
| $prgRV$ | 0.62 | 0.95 | −8003.15 | −3868.28 | 0.66 | 1.00 | −4197.41 | −2215.86 | 0.69 | 1.06 | −4160.21 | −2210.63 |
| $prgQRV$ | 0.72 | 0.96 | −7134.90 | −3609.96 | 0.52 | 1.03 | −5054.52 | −2524.47 | 0.73 | 1.18 | −4931.84 | $NaN$ |
| $Average$ | 0.63 | 0.95 | −7935.80 | −3858.45 | 0.65 | 1.00 | −4207.03 | −2258.64 | 0.70 | 1.07 | −4188.51 | $NaN$ |

Table 9 provides estimation results for FloGARCH(1,d,1), FloLGARCH(1,d,1) and FloEGARCH(1,d,1) estimated on the whole data set of stocks using different realized measures. It summarizes results for $r_t^{cc}$ and is the twin Table of Table 9, which provided results for $r_t^{oc}$. For each model, the value of the parameters $d$ and $\varphi$ are reported. The parameter $d$ summarizes the level of memory in the model while $\varphi$ is informative of potential biases contained in realized measures. The in-sample and the out-of sample likelihood was computed following the procedure described in Section 3 devoted to QMLE.

Table 10: Comparison Table for constrained versions of FloGARCH - $r_t^{cc}$.

| | FloLin | | | | FloLog | | | | FloExp | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ | $l^{IS}(r)$ | $l^{OS}(r)$ | $AIC(p)$ | $BIC(p)$ |
| *Panel A: Unrestricted Models* | | | | | | | | | | | | |
| *Average* | −8251.2 | −3781.4 | 16520.4 | 16635.4 | −4167.9 | −2119.1 | 8353.8 | 8468.8 | −4140.7 | −2116.7 | 8303.5 | 8444.0 |
| *Median* | −7947.5 | −3796.6 | 15912.9 | 16027.9 | −4044.3 | −2136.3 | 8106.6 | 8221.7 | −4003.5 | −2134.7 | 8029.1 | 8169.6 |
| $Q_{.25}$ | −9151.9 | −4490.4 | 14482.0 | 14597.0 | −4607.4 | −2347.1 | 7563.7 | 7678.7 | −4576.9 | −2360.9 | 7493.3 | 7633.9 |
| $Q_{.75}$ | −7232.0 | −2906.4 | 18321.8 | 18436.8 | −3772.8 | −1813.5 | 9232.9 | 9347.9 | −3735.6 | −1812.2 | 9175.7 | 9316.3 |
| *Panel B:* $\mu = 0$ | | | | | | | | | | | | |
| *Average* | −8251.9 | −3781.4 | 16519.9 | 16622.1 | −4168.7 | −2120.2 | 8353.3 | 8455.6 | −4141.6 | −2115.7 | 8303.2 | 8431.0 |
| *Median* | −7947.9 | −3797.6 | 15911.7 | 16014.0 | −4045.8 | −2136.6 | 8107.6 | 8209.8 | −4003.6 | −2134.6 | 8027.2 | 8155.0 |
| $Q_{.25}$ | −9152.6 | −4489.9 | 14481.4 | 14583.6 | −4608.3 | −2350.3 | 7561.9 | 7664.1 | −4578.7 | −2358.1 | 7491.4 | 7619.2 |
| $Q_{.75}$ | −7232.7 | −2906.6 | 18321.1 | 18423.4 | −3772.9 | −1813.2 | 9232.5 | 9334.8 | −3735.7 | −1811.5 | 9177.3 | 9305.1 |
| *Panel C:* $\varphi = 1$ | | | | | | | | | | | | |
| *Average* | −8253.6 | −3780.6 | 16523.2 | 16625.4 | −4170.0 | −2118.3 | 8356.0 | 8458.3 | −4145.9 | −2117.0 | 8311.8 | 8439.6 |
| *Median* | −7948.6 | −3799.2 | 15913.1 | 16015.3 | −4044.4 | −2137.9 | 8104.8 | 8207.1 | −4004.4 | −2134.0 | 8028.8 | 8156.6 |
| $Q_{.25}$ | −9154.8 | −4491.1 | 14481.1 | 14583.3 | −4610.7 | −2348.5 | 7563.0 | 7665.2 | −4594.9 | −2360.3 | 7500.7 | 7628.5 |
| $Q_{.75}$ | −7232.6 | −2907.0 | 18325.6 | 18427.9 | −3773.5 | −1813.5 | 9237.4 | 9339.6 | −3740.4 | −1812.4 | 9209.8 | 9337.6 |
| *Panel D:* $\mu = 0 \cap \varphi = 1$ | | | | | | | | | | | | |
| *Average* | −8254.3 | −3780.5 | 16522.7 | 16612.1 | −4170.6 | −2118.1 | 8355.2 | 8444.7 | −4146.9 | −2116.1 | 8311.7 | 8426.7 |
| *Median* | −7948.9 | −3800.2 | 15911.9 | 16001.4 | −4045.9 | −2138.0 | 8105.8 | 8195.2 | −4004.4 | −2134.0 | 8026.9 | 8141.9 |
| $Q_{.25}$ | −9155.5 | −4490.7 | 14480.5 | 14570.0 | −4610.1 | −2348.2 | 7561.2 | 7650.7 | −4596.3 | −2357.7 | 7498.8 | 7613.8 |
| $Q_{.75}$ | −7233.3 | −2907.3 | 18325.0 | 18414.5 | −3773.6 | −1813.3 | 9234.3 | 9323.7 | −3740.4 | −1811.6 | 9210.6 | 9325.6 |

Table 10 reports sample means of the statistics computed over the 65 stocks available in the data set. The results rely on $r_t^{cc}$ and the same specification is used for the three classes of models: (1,d,1).

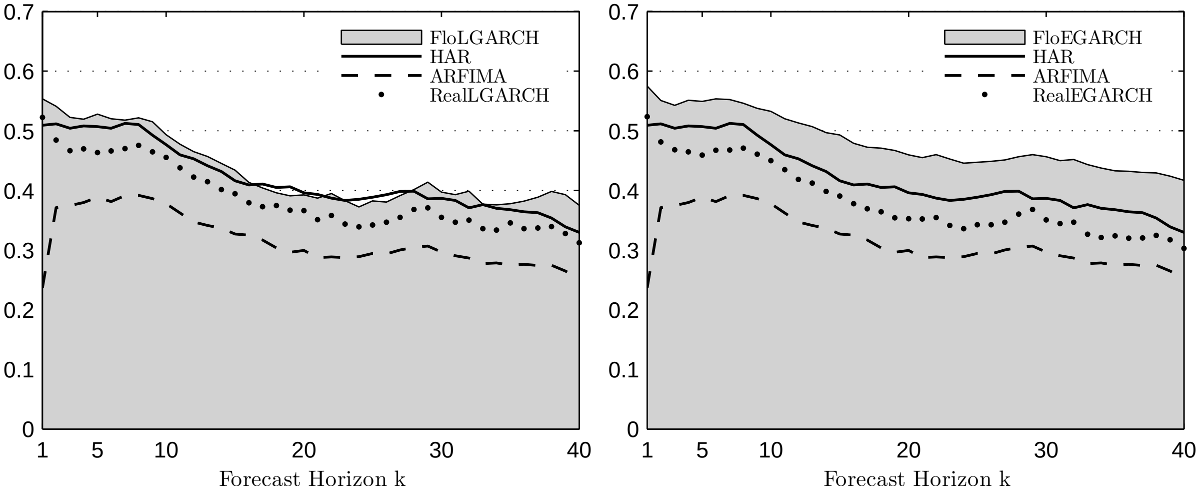


Figure 7: Mincer-Zarnowitz $R^2$ for FloLGARCH(1,d,1) and FloEGARCH(1,d,1) computed from $RV^{5min}$.

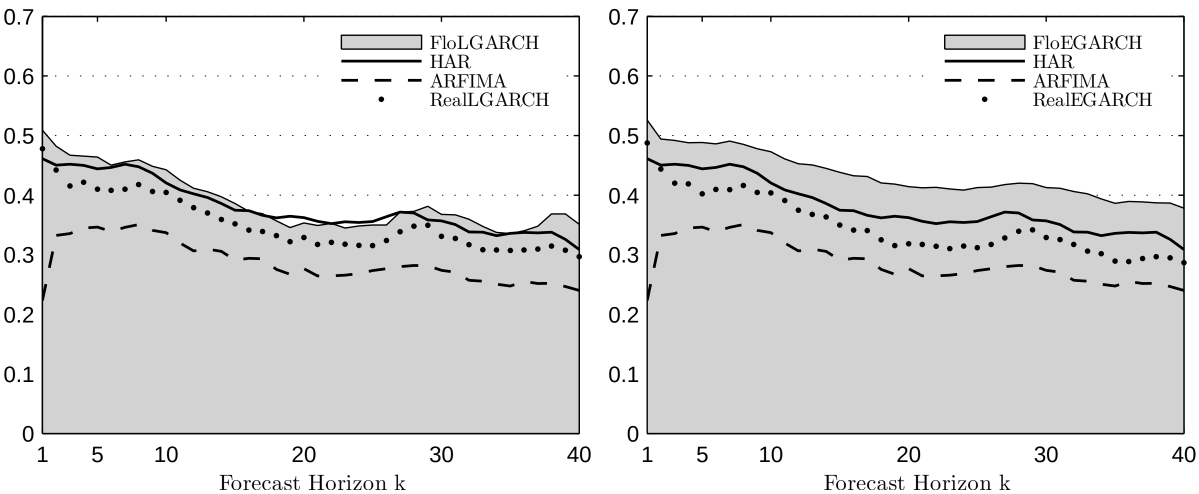


Figure 8: Mincer-Zarnowitz $R^2$ for FloLGARCH(1,d,1) and FloEGARCH(1,d,1) computed from $ssRV^{15min}$.

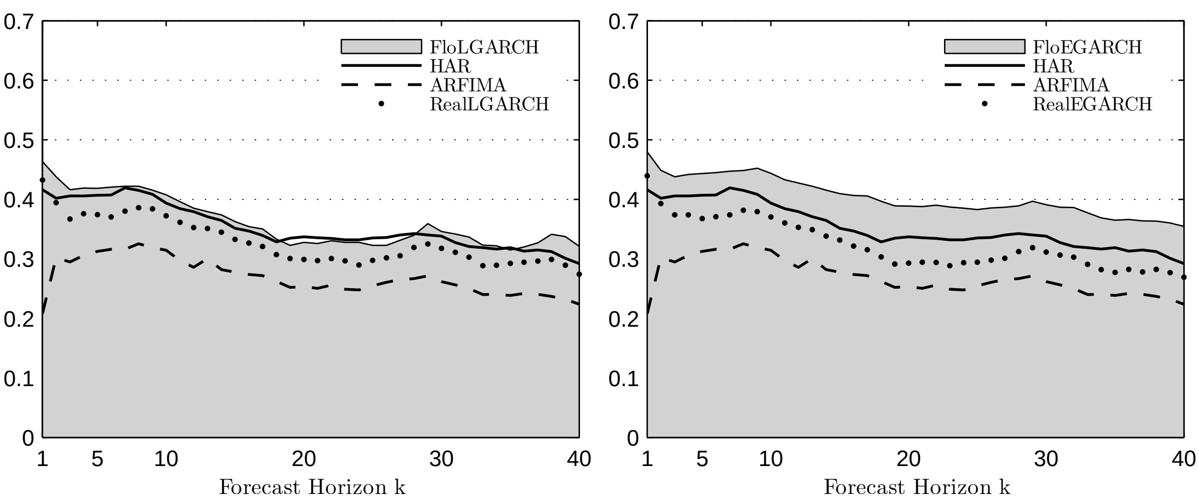


Figure 9: Mincer-Zarnowitz $R^2$ for FloLGARCH(1,d,1) and FloEGARCH(1,d,1) computed from $RV^{15min}$.

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