Tractable Term Structure Models and the Zero Lower Bound

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National Bank of Belgium
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Motivation

One-month interest rate

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Consider the Gaussian models:

\[ M_{t+1} = e^{-\left(\delta_0 + \delta_1' X_t\right)} e^{-\frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \epsilon_{t+1}} \]

where

\[ X_{t+1} = K_0 + K_1 X_t + \Sigma \epsilon_{t+1}, \]

\[ \epsilon_{t+1} \sim N(0, I), \]

\[ \lambda_t = \lambda_0 + \lambda_1 X_t, \]

\[ r_t \approx 0, \]

are tractable:

\[ y_{nt}, \]

but problematic:

\[ r_{t+1} \]

because when \( r_t \approx 0 \), model says 50% chance \( r_{t+1} \) will be negative.

Consider the Black's models:

\[ M_{t+1} = e^{-\max \left(\delta_0 + \delta_1' X_t, 0\right)} e^{-\frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \epsilon_{t+1}} \]

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  $$y_{n,t} \geq 0$$
Motivation

More generally, in constructing no-arbitrage term structure models, we are often constrained by tractability considerations:

- We tend to focus on a subset of sdfs $M_t > 0$ such that:

  $$P_{1,t} = E_t[M_{t+1}] \text{ is closed form},$$
  $$P_{2,t} = E_t[M_{t+1}M_{t+2}] \text{ is closed form},$$
  $$\ldots,$$
  $$P_{n,t} = E_t[M_{t+1}M_{t+2}\ldots M_{t+n}] \text{ is closed form}$$

- Focusing on this subset of sdf’s may restrict our ability to explore more realistic models
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**Question:** Can we explore more realistic models yet maintaining tractability in pricing?
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- Instead, we specify bond prices directly:

\[
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P_{2,t} = p_2(X_t), \text{ for some analytical function } p_2(.) \\
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- Importantly, we choose $p_1, p_2, ..., p_n$ such that we come VERY close to ruling out arbitrage opportunities
Summary of our approach

- **Our approach is highly flexible:** researchers have complete freedom in specifying the one-period bond price:

  \[ P_{1,t} = p_1(X_t) \]

- Imposing a lower bound on the short rate is straightforward: choose \[ P_{1,t} = p_1(X_t) < 1 \] for all \( X_t \)

- We can generate a wide range of tractable nonlinear term structure models
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- The key intuition: with certain parameterizations, we can "undo" the nonlinearity and translate our models back to the linear space:

\[ f_{\text{linear}}(t) = \left( A_n + B_n X_t \right)^3 \rightarrow \left( f_{\text{linear}}(t) \right)^{1/3} = A_n + B_n X_t. \]
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- **The key intuition:** with certain parameterizations, we can ”undo” the nonlinearity and translate our models back to the linear space:
  - For example, for the sake of intuition:
    \[ f_{n,t} = (A_n + B_nX_t)^3 \rightarrow f_{n,t}^{1/3} = A_n + B_nX_t. \]
    We can proceed with estimation of a linear model as in JSZ with the observed data being \( f_t^{1/3} \).
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    We can proceed with estimation of a linear model as in JSZ with the observed data being \( f_t^{1/3} \).
  - More generally, under certain reasonable parameterizations, we have:
    \[
    f_{n,t} = m(A_n + B_n X_t) \rightarrow m^{-1}(f_{n,t}) = A_n + B_n X_t.
    \]
    The outcome: instantaneous convergence with guaranteed global estimates.
Outline

1. Our construction of bond prices
2. How close are we to no arbitrage?
3. Closed form yields and forwards
4. Examples
5. Time series dynamics
6. Empirical Illustrations
1. Our construction of bond prices

- The n-period zero-coupon bond price $P_n$ is given recursively by

\[
P_0(X_t) \equiv 1, \tag{1}
\]
\[
P_n(X_t) = P_{n-1}(g(X_t)) \times \exp(-m(X_t)), \tag{2}
\]

for some functions $m(\cdot)$ and $g(\cdot)$.
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for some functions $m(\cdot)$ and $g(\cdot)$.

- Example 1:

$$P_1(X_t) = P_0(g(X_t)) \times \exp(-m(X_t)) = \exp(-m(X_t))$$ \hspace{1cm} (3)

- $m(\cdot)$ gives the one-period rate
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- Example 2:

\[
P_2(X_t) = P_1(g(X_t)) \times \exp(-m(X_t))
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\[
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▶ $g(\cdot)$ allows us to go recursively from $P_{n-1}(\cdot)$ to $P_n(\cdot)$. 
2. How close are we to no arb?

Two relevant concepts:

1. **No dominant trading strategies:**
   a portfolio with **strictly positive** payoffs must have a strictly positive price
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1. **No dominant trading strategies:**
   a portfolio with strictly positive payoffs must have a strictly positive price

2. **No arbitrage opportunities:**
   a portfolio with non-negative payoffs must have a strictly positive price
   - non-negative payoffs $\equiv$ strictly positive payoffs for some positive probability and zero payoffs otherwise
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Theorem 1: Our bond price construction allows no dominant trading strategies
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**Theorem 1:** Our bond price construction allows no dominant trading strategies

**Proof.**

1. Consider a portfolio: investment in $n$-period bond is $w_n$ (in face value)

\[
\text{Price of portfolio} = \sum w_n P_n(X_t), \quad (5)
\]

\[
= \sum w_n \exp(-m(X_t)) P_{n-1}(g(X_t)), \quad (6)
\]

\[
\geq 0 \quad (7)
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   - the price this period: $\sum_n w_n P_n(X_t)$,
   - the payoff next period: $\sum_n w_n P_{n-1}(X_{t+1}) > 0$ for all $X_{t+1} \in X$

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   \(> 0\)
2. How close we are to ruling out no arb opportunities?

**Theorem 2:** Our bond price construction ensures that bond portfolios with strictly non-negative payoffs cannot admit strictly negative prices.

*Figure:* Prices of portfolios with strictly non-negative payoffs
3. Closed form yields and forwards

The n-period yields and forward rates are given by

\[ y_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} m(g^{\circ i}(X_t)) \quad (8) \]

\[ f_{n,t} \equiv (n + 1)y_{n+1,t} - ny_{n,t} = m(g^{\circ n}(X_t)). \quad (9) \]

\[ g^{\circ n}(X_t) \] denotes \[ g(g(....g(X_t)...)) \] (n times),
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- \(g^{\circ n}(X_t)\) denotes \(g(g(\ldots g(X_t)\ldots))\) \((n\ \text{times})\),

- For example, if \(g(X_t) = K_1X_t\), then \(g^{\circ n}(X_t) = K_1^nX_t\). With this choice:

\[
f_{n,t} = m(K_1^nX_t),
\]

thus the nonlinearity can be “undone” by inverting the \(m(\cdot)\) function.
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\[ \Rightarrow \]

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This is (essentially) the Nelson-Siegel model!

Compare with the standard Gaussian no-arbitrage DTSM:

\[ nA_n = (n - 1)A_{n-1} + \delta_0 - \frac{1}{2}B_{n-1}\Sigma B'_{n-1} \]

\[ B_n = B_{n-1}K^Q_1 + \delta'_1. \tag{12} \]
4. Examples: the Nelson-Siegel model

**Proposition 1:** Suppose $X_t \in \mathbb{R}^3$, with $m(\cdot)$ and $g(\cdot)$ given by

\[
m(X_t) = \begin{bmatrix} 1 & \frac{1-e^{-\lambda}}{\lambda} & \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda} \end{bmatrix} X_t, \tag{13}
\]

\[
g(X_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{bmatrix} X_t, \tag{14}
\]

then, our bond prices construction implies yields with Nelson-Siegel loadings.

**Proof.**

Direct computation of $y_{t,n} = (1/n) \sum_{i=0}^{n-1} m(g^{\circ i})$ yields the result.
4. The Nelson-Siegel model

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- Despite this short-coming, the N-S model has been highly popular!
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  - Theoretically, Krippner (2013) show that the N-S model can be seen as a low-order Taylor approximation of certain no-arb Gaussian affine models
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- Our analysis focuses on no-dominance trading strategies and is applicable more generally.
4. Examples: the linear-quadratic case

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\[ g(X) = KX \quad m(X) = \delta_0 + \delta'_1 X + X'\delta_2 X \]  \quad (15)

\[ \Rightarrow \text{quadratic yield coefficients: } y_{n,t} = A_n + (B_n/n)X_t + X'_t(C_n/n)X_t \]

\[ A_n = \delta_0 \]
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Compare with the standard Gaussian-quadratic no-arbitrage DTSM:

\[ nA_n = (n - 1)A_{n-1} + \delta_0 - \frac{1}{2} \log|\Omega_{n-1}| - \frac{1}{2} B_{n-1} \Omega_{n-1} \Sigma B_{n-1} \]
\[ B_n = B_{n-1} \Omega_{n-1} K_1^Q + \delta'_1 \]
\[ C_n = K_1^{Q'} C_{n-1} \Omega_{n-1} K_1^Q + \delta_2, \]  \[ (17) \]

with \( \Omega_{n-1} \equiv (I_N - 2\Sigma C_{n-1})^{-1}. \)
4. Examples: Black-style models

- Recall that the main idea here is to guarantee the positivity of yields.
- In Black’s models, this is achieved by \( r_t = \max(0, \delta_0 + \delta'_1 X_t) \).
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- Recall that the main idea here is to guarantee the positivity of yields.
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Within our framework:

- We choose the same \( g(X) = KX \)
- We choose a generalized logistic transformation for the short rate function:

\[
m(X) = \theta \log \left( 1 + \exp \left( \frac{\delta_0 + \delta'_1 X}{\theta} \right) \right),
\]

- We can think about \( \delta_0 + \delta'_1 X_t \) as a shadow rate which can be negative but the short rate is always positive after the \( m(.) \) transformation.
4. Examples: Black-style models

Our choice of $m(X)$ captures the spirit of the $\max(0, \delta_0 + \delta_1 X_t)$ transformation (to guarantee positivity) in Black’s models:
4. Examples: Black-style models

- We deliver analytical yields/ forwards:
  \[
  f_{n,t} = u(\theta, \delta_0 + \delta_1 K^n X_t) \tag{18}
  \]
  where \( u(\theta, s) \) captures the logistic transformation:
  \[
  u(\theta, s) = \theta \log(1 + \exp(s/\theta)).
  \]

- This means that we can work with transformed forwards
  \[
  \tilde{f}_{n,t} \equiv u^{-1}(\theta, f_{n,t}) = \delta_0 + \delta_1 K^n X_t. \tag{19}
  \]
  and we are back to the linear space!
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  - Unspanned factors
  - Long or infinite lag structure
  - Shifting endpoints and unit roots.
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  - Shifting endpoints and unit roots.
6. Empirical Illustrations

- We focus on the no-dominance (ND) versions of two Black’s style models:
  - $Black_{ND}$: A model in which the states follow a Gaussian VAR(1). This is a close analogue to the traditional Black’s model.
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- All models considered are three-factor models.
- Analysis is done in two ways: 1) through a simulated environment; 2) using the US yields data;
6. Empirical Illustrations – a Simulated Environment

We use the linear rational model of Filipovic, Larson, and Trolle (JF 2017) as a DGP to generate 100 samples of yields data that exhibit:

- Salient features of the yield data in the U.S., such as bond returns predictability, the shapes of the yield curve, time-varying volatility etc.
- Binding ZLB

Our plan is straightforward:

- We estimate the four models under consideration using each of the 100 data samples
- We then compare the model-implied forecasts (yield, volatility, Sharpe ratio) to the true forecasts

Main advantage of a simulation environment:

- We do know the true forecasts
6. Empirical Illustrations – a Simulated Environment

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  - We do know the **true forecasts**
6. Empirical Illustrations – a Simulated Environment

(a) Simulated yields

(b) Average Means and Vols of Simulated Yields

**Figure:** Statistics of simulated yields. Sample #1
### Table: Bond Yield Forecast Errors

The symbol * indicates the best performance for each forecast horizon $h$ and yield maturity $mat$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$mat$</th>
<th>$Gaussian_{NA}$</th>
<th>$Black_{NA}$</th>
<th>$Black_{ND}$</th>
<th>$SV-Black_{ND}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-m</td>
<td>1-m</td>
<td>10.8</td>
<td>0.91</td>
<td>0.96</td>
<td>0.75*</td>
</tr>
<tr>
<td>3-m</td>
<td>2-yr</td>
<td>8.0</td>
<td>0.96</td>
<td>0.93</td>
<td>0.81*</td>
</tr>
<tr>
<td></td>
<td>5-yr</td>
<td>7.9</td>
<td>1.00</td>
<td>0.90</td>
<td>0.82*</td>
</tr>
<tr>
<td></td>
<td>10-yr</td>
<td>7.8</td>
<td>1.00</td>
<td>0.97</td>
<td>0.75*</td>
</tr>
<tr>
<td>1-yr</td>
<td>1-m</td>
<td>27.5</td>
<td>0.94</td>
<td>0.92</td>
<td>0.82*</td>
</tr>
<tr>
<td></td>
<td>2-yr</td>
<td>26.1</td>
<td>0.97</td>
<td>0.91</td>
<td>0.77*</td>
</tr>
<tr>
<td></td>
<td>5-yr</td>
<td>27.8</td>
<td>1.00</td>
<td>0.91</td>
<td>0.72*</td>
</tr>
<tr>
<td></td>
<td>10-yr</td>
<td>26.9</td>
<td>1.01</td>
<td>0.96</td>
<td>0.75*</td>
</tr>
</tbody>
</table>
6. Empirical Illustrations – a Simulated Environment

<table>
<thead>
<tr>
<th>h</th>
<th>mat</th>
<th>( Gaussian_{NA} )</th>
<th>( Black_{NA} )</th>
<th>( Black_{ND} )</th>
<th>( SV-Black_{ND} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-m</td>
<td>7.8</td>
<td>0.69</td>
<td>0.58*</td>
<td>0.69</td>
<td></td>
</tr>
<tr>
<td>2-yr</td>
<td>7.5</td>
<td>0.76</td>
<td>0.49*</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>5-yr</td>
<td>7.2</td>
<td>1.01</td>
<td>0.74</td>
<td>0.62*</td>
<td></td>
</tr>
<tr>
<td>10-yr</td>
<td>6.8</td>
<td>0.91</td>
<td>0.91</td>
<td>0.65*</td>
<td></td>
</tr>
<tr>
<td>1-m</td>
<td>24.0</td>
<td>0.67</td>
<td>0.47*</td>
<td>0.74</td>
<td></td>
</tr>
<tr>
<td>2-yr</td>
<td>23.4</td>
<td>0.87</td>
<td>0.66</td>
<td>0.54*</td>
<td></td>
</tr>
<tr>
<td>5-yr</td>
<td>24.1</td>
<td>0.97</td>
<td>0.81</td>
<td>0.61*</td>
<td></td>
</tr>
<tr>
<td>10-yr</td>
<td>21.3</td>
<td>1.00</td>
<td>1.04</td>
<td>0.70*</td>
<td></td>
</tr>
</tbody>
</table>

Table: **Bond Yield Forecast Errors** The symbol * indicates the best performance for each forecast horizon \( h \) and yield maturity \( mat \).

---
6. Empirical Illustrations – a Simulated Environment

<table>
<thead>
<tr>
<th>h</th>
<th>mat</th>
<th>Full-Sample</th>
<th></th>
<th></th>
<th></th>
<th>ZLB Sample</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$G_{NA}$</td>
<td>$B_{NA}$</td>
<td>$B_{ND}$</td>
<td>SV-$B_{ND}$</td>
<td>$G_{NA}$</td>
<td>$B_{NA}$</td>
<td>$B_{ND}$</td>
<td>SV-$B_{ND}$</td>
</tr>
<tr>
<td>1-m</td>
<td>3-m</td>
<td>54.3</td>
<td>0.30</td>
<td>0.51</td>
<td>0.29*</td>
<td>68.9</td>
<td>0.15</td>
<td>0.14*</td>
<td>0.15</td>
</tr>
<tr>
<td>2-yr</td>
<td>5-yr</td>
<td>30.9</td>
<td>0.50</td>
<td>0.63</td>
<td>0.34*</td>
<td>36.8</td>
<td>0.34</td>
<td>0.18*</td>
<td>0.20</td>
</tr>
<tr>
<td>5-yr</td>
<td>10-yr</td>
<td>19.4</td>
<td>0.76</td>
<td>0.74</td>
<td>0.48*</td>
<td>25.2</td>
<td>0.42</td>
<td>0.42</td>
<td>0.29*</td>
</tr>
<tr>
<td>10-yr</td>
<td>1-yr</td>
<td>15.0</td>
<td>0.62</td>
<td>0.94</td>
<td>0.59*</td>
<td>20.1</td>
<td>0.28*</td>
<td>0.76</td>
<td>0.45</td>
</tr>
</tbody>
</table>

| 1-m | 1-yr| 63.6        | 0.72    | 0.53    | 0.42*   | 67.0       | 0.67    | 0.34*   | 0.38    |
| 2-yr|     | 39.9        | 1.04    | 0.61    | 0.46*   | 39.8       | 1.07    | 0.39*   | 0.40    |
| 5-yr|     | 27.9        | 1.20    | 0.63    | 0.58*   | 29.8       | 1.16    | 0.49    | 0.44*   |
| 10-yr|    | 18.6        | 1.23    | 0.86    | 0.76*   | 22.3       | 0.93    | 0.69    | 0.56*   |

**Table:** Bond Yield Volatility Forecast Errors The symbol * indicates the best performance.
6. Empirical Illustrations – a Simulated Environment

<table>
<thead>
<tr>
<th>h</th>
<th>mat</th>
<th>Full-Sample</th>
<th>ZLB Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$G_{NA}$</td>
<td>$B_{NA}$</td>
</tr>
<tr>
<td>3-m</td>
<td>6-m</td>
<td>0.2</td>
<td>1.62</td>
</tr>
<tr>
<td></td>
<td>2-yr</td>
<td>0.5</td>
<td>1.59</td>
</tr>
<tr>
<td></td>
<td>5-yr</td>
<td>0.8</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td>9-yr</td>
<td>1.2</td>
<td>1.24</td>
</tr>
<tr>
<td>1-yr</td>
<td>6-m</td>
<td>0.2</td>
<td>1.59</td>
</tr>
<tr>
<td></td>
<td>2-yr</td>
<td>0.3</td>
<td>1.57</td>
</tr>
<tr>
<td></td>
<td>5-yr</td>
<td>0.6</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td>9-yr</td>
<td>0.9</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table: Bond Sharpe Ratio Forecast Errors The symbol * indicates the best performance.
### 6. Empirical Illustrations – a Simulated Environment

<table>
<thead>
<tr>
<th>h</th>
<th>mat</th>
<th>$G_{NA}$</th>
<th>$B_{NA}$</th>
<th>$B_{ND}$</th>
<th>SV-$B_{ND}$</th>
<th>$G_{NA}$</th>
<th>$B_{NA}$</th>
<th>$B_{ND}$</th>
<th>SV-$B_{ND}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6m-9y</td>
<td>0.5</td>
<td>1.63</td>
<td>1.01</td>
<td>0.77*</td>
<td></td>
<td>0.7</td>
<td>1.04</td>
<td>0.49</td>
<td>0.45*</td>
</tr>
<tr>
<td>3-m</td>
<td>0.8</td>
<td>1.46</td>
<td>0.98</td>
<td>0.74*</td>
<td></td>
<td>1.0</td>
<td>1.10</td>
<td>0.73</td>
<td>0.49*</td>
</tr>
<tr>
<td>5y-9y</td>
<td>1.0</td>
<td>1.33</td>
<td>0.93</td>
<td>0.71*</td>
<td></td>
<td>1.2</td>
<td>1.04</td>
<td>0.76</td>
<td>0.52*</td>
</tr>
<tr>
<td>6m-9y</td>
<td>0.4</td>
<td>1.61</td>
<td>1.01</td>
<td>0.76*</td>
<td></td>
<td>0.4</td>
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<td>0.97</td>
<td>0.64*</td>
<td></td>
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<td>1.27</td>
<td>0.95</td>
<td>0.58*</td>
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**Table:** Bond Portfolio Sharpe Ratio Forecast Errors The symbol * indicates the best performance.

Figure: U.S. 1-month rate and 1-month shadow rate over sample period 1970:Jan - 2015:Dec

Figure: U.S. 1-month rate and 1-month shadow rate over sample period 2008:Jan - 2015:Dec

Figure: Forecasts of 1-m yields over sample period 2008:Jan - 2015:Dec

(a) 3-month forecast horizon

(b) 12-month forecast horizon

Figure: Volatility forecasts of 120-m yields over sample period 1970:Jan - 2015:Dec

<table>
<thead>
<tr>
<th></th>
<th>Entire Sample</th>
<th>ZLB Sample</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$B_{NA}$</td>
<td>$B_{ND}$</td>
</tr>
<tr>
<td>3-m Average</td>
<td>25.02</td>
<td>0.75</td>
</tr>
<tr>
<td>12-m Average</td>
<td>45.81</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Table: RMSE (in basis points) btw RV and volatility forecasts

<table>
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<th></th>
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<td>$SV-B_{ND}$</td>
<td>$B_{NA}$</td>
</tr>
<tr>
<td>3-m Average</td>
<td>0.26</td>
<td>0.25</td>
<td>0.79*</td>
<td>0.67</td>
</tr>
<tr>
<td>12-m Average</td>
<td>0.25</td>
<td>0.24</td>
<td>0.79*</td>
<td>0.59</td>
</tr>
</tbody>
</table>

Table: Correlation between volatility forecasts and RV

Figure: EH component of 120-m yields over sample period 1970:Jan - 2015:Dec

Figure: EH component of 120-m yields, relative to Gaussian model, over sample period 1970:Jan - 2015:Dec
Propose an alternative approach to designing new tractable term structure models:

1. We specify bond prices directly without going through any sdf.
2. We then verify that these prices are free of dominant trading strategies.

Imposition of lower bounds on yields is straightforward yet maintaining tractability and ease of implementation

Simulation exercises show that our models can be at least comparable to some existing popular ZLB models