Tractable Term Structure Models and the Zero Lower Bound

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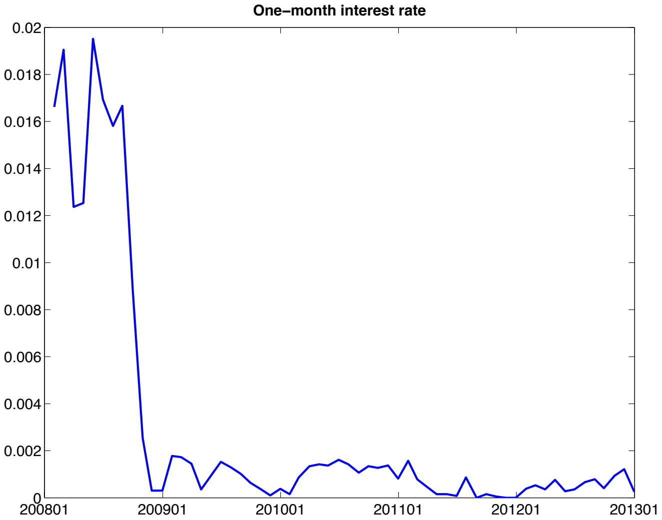
Motivation



One−month interest rate
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
200801
200901
201001
201101
201201
201301

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Consider the Gaussian models:

$$M_{t+1} = e^{-\underbrace{(\delta_0 + \delta_1' X_t)}_{r_t} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\epsilon_{t+1}},$$

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$$X_{t+1} = K_0 + K_1 X_t + \Sigma\epsilon_{t+1},$$

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$y_{n,t}$'s are no longer closed-form

- **but receive more attention:**

$$y_{n,t} \geq 0$$

Motivation

- More generally, in constructing no-arbitrage term structure models, we are often constrained by tractability considerations:
  - We tend to focus on a subset of sdfs $M_t > 0$ such that:

$$\begin{aligned} P_{1,t} &= E_t[M_{t+1}] \text{ is closed form,} \\ P_{2,t} &= E_t[M_{t+1} M_{t+2}] \text{ is closed form,} \\ &\ldots, \\ P_{n,t} &= E_t[M_{t+1} M_{t+2} \ldots M_{t+n}] \text{ is closed form} \end{aligned}$$

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- **Question: Can we explore more realistic models yet maintaining tractability in pricing?**

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- **Importantly, we choose $p_1, p_2, ..., p_n$ such that we come VERY close to ruling out arbitrage opportunities**

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  - We can generate a wide range of tractable nonlinear term structure models

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    We can proceed with estimation of a linear model as in JSZ with the observed data being $f_t^{1/3}$.
  - More generally, under certain reasonable parameterizations, we have:

    $$f_{n,t} = m(A_n + B_n X_t) \rightarrow m^{-1}(f_{n,t}) = A_n + B_n X_t.$$

    The outcome: instantaneous convergence with guaranteed global estimates.

Outline

1. Our construction of bond prices
2. How close are we to no arbitrage?
3. Closed form yields and forwards
4. Examples
5. Time series dynamics
6. Empirical Illustrations

1. Our construction of bond prices

- The n-period zero-coupon bond price P_n is given recursively by

$$P_0(X_t) \equiv 1, \quad (1)$$

$$P_n(X_t) = P_{n-1}(g(X_t)) \times exp(-m(X_t)), \quad (2)$$

for some functions $m(\cdot)$ and $g(\cdot)$.

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- Example 1:

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- Example 2:

$$\begin{aligned} P_2(X_t) &= P_1(g(X_t)) \times exp(-m(X_t)) \\ &= exp(-m(g(X_t))) \times exp(-m(X_t)) \quad (4) \end{aligned}$$

  - $g(\cdot)$ allows us to go recursively from $P_{n-1}(\cdot)$ to $P_n(\cdot)$.

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Two relevant concepts:

1. **No dominant trading strategies:**
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1. **No dominant trading strategies:**
   a portfolio with strictly positive payoffs must have a strictly positive price

2. **No arbitrage opportunities:**
   a portfolio with non-negative payoffs must have a strictly positive price
   - non-negative payoffs $\equiv$ strictly positive payoffs for some positive probability and zero payoffs otherwise

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$$\text{Price of portfolio} = \sum_n w_n P_n(X_t), \tag{5}$$

$$= \sum_n w_n exp(-m(X_t)) P_{n-1}(g(X_t)), \tag{6}$$

$$= exp(-m(X_t)) \underbrace{\sum_n w_n P_{n-1}(g(X_t))}_{>0} \tag{7}$$

2. How close we are to ruling out no arb opportunities?

Theorem 2: Our bond price construction ensures that bond portfolios with strictly non-negative payoffs cannot admit strictly negative prices.

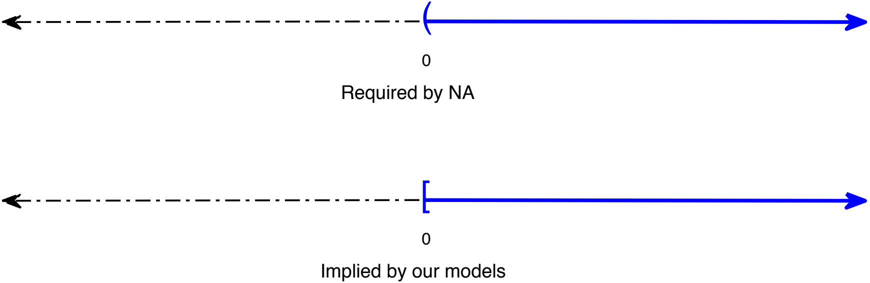

Figure: Prices of portfolios with strictly non-negative payoffs

3. Closed form yields and forwards

- The n-period yields and forward rates are given by

$$y_{n,t} = (1/n) \sum_{i=0}^{n-1} m(g^{\circ i}(X_t)) \quad (8)$$

$$f_{n,t} \equiv (n+1) y_{n+1,t} - n y_{n,t} = m(g^{\circ n}(X_t)). \quad (9)$$

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  - $g^{\circ n}(X_t)$ denotes $g(g(....g(X_t)...))$ (n times),
  - For example, if $g(X_t) = K_1 X_t$, then $g^{\circ n}(X_t) = K_1^n X_t$. With this choice:

$$f_{n,t} = m(K_1^n X_t),$$

    thus the nonlinearity can be "undone" by inverting the $m(\cdot)$ function.

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with

$$\begin{aligned} A_n =& \delta_0 \\ B_n =& B_{n-1}K + \delta_1'. \end{aligned} \tag{11}$$

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Compare with the standard Gaussian no-arbitrage DTSM:

$$\begin{aligned} nA_n =& (n-1)A_{n-1} + \delta_0 - \frac{1}{2} B_{n-1} \Sigma B_{n-1}' \\ B_n =& B_{n-1} K_1^{\mathbb{Q}} + \delta_1'. \end{aligned} \tag{12}$$

4. Examples: the Nelson-Siegel model

Proposition 1: Suppose $X_t \in \mathbb{R}^3$, with $m(\cdot)$ and $g(\cdot)$ given by

$$m(X_t) = \begin{bmatrix} 1 & \frac{1-e^{-\lambda}}{\lambda} & \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda} \end{bmatrix} X_t, \qquad (13)$$

$$g(X_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{bmatrix} X_t, \qquad (14)$$

then, our bond prices construction implies yields with Nelson-Siegel loadings.

Proof.

Direct computation of $y_{t,n} = (1/n) \sum_{i=0}^{n-1} m(g^{\circ i})$ yields the result. □

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  - Theoretically, Krippner (2013) show that the N-S model can be seen as a low-order Taylor approximation of certain no-arb Gaussian affine models
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- **Our analysis focuses on no-dominance trading strategies and is applicable more generally.**

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Compare with the standard Gaussian-quadratic no-arbitrage DTSM:

$$\begin{aligned} nA_n &= (n-1)A_{n-1} + \delta_0 - \frac{1}{2} log|\Omega_{n-1}| - \frac{1}{2} B_{n-1}\Omega_{n-1}\Sigma B_{n-1} \\ B_n &= B_{n-1}\Omega_{n-1}K_1^{\mathbb{Q}} + \delta_1' \\ C_n &= K_1^{\mathbb{Q}\prime} C_{n-1}\Omega_{n-1}K_1^{\mathbb{Q}} + \delta_2, \end{aligned} \tag{17}$$

with $\Omega_{n-1} \equiv (I_N - 2\Sigma C_{n-1})^{-1}$.

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Within our framework:

- We choose the same $g(X) = KX$
- We choose a generalized logistic transformation for the short rate function:

$$m(X) = \theta log \left(1 + exp\left(\frac{\delta_0 + \delta_1' X}{\theta}\right)\right),$$

- We can think about $\delta_0 + \delta_1' X_t$ as a shadow rate which can be negative but the short rate is always positive after the $m(.)$ transformation

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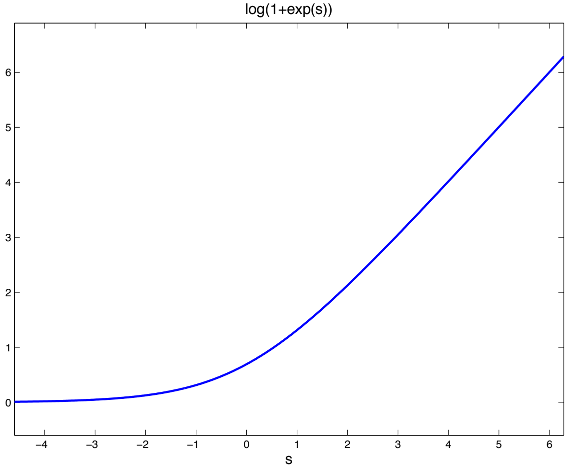
Our choice of $m(X)$ captures the spirit of the $max(0, \delta_0 + \delta_1' X_t)$ transformation (to guarantee positivity) in Black's models:



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- We deliver analytical yields/ forwards:

$$f_{n,t} = u(\theta, \delta_0 + \delta_1' K^n X_t) \tag{18}$$

where $u(\theta, s)$ captures the logistic transformation:
$u(\theta, s) = \theta log(1 + exp(s/\theta))$.

- This means that we can work with transformed forwards

$$\tilde{f}_{n,t} \equiv u^{-1}(\theta, f_{n,t}) = \delta_0 + \delta_1' K^n X_t. \tag{19}$$

and we are back to the linear space!

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6. Empirical Illustrations

- We focus on the no-dominance (ND) versions of two Black's style models:
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- For comparison purposes, we consider two no-arbitrage models:
  1. $Gaussian_{NA}$: the standard no-arbitrage affine Gaussian term structure models
  2. $Black_{NA}$: the traditional no-arbitrage Black's model
- All models considered are three-factor models.

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  2. SV-$Black_{ND}$: A model in which the states dynamics features stochastic volatility (SV) (in a GARCH style).
- For comparison purposes, we consider two no-arbitrage models:
  1. $Gaussian_{NA}$: the standard no-arbitrage affine Gaussian term structure models
  2. $Black_{NA}$: the traditional no-arbitrage Black's model
- All models considered are three-factor models.
- Analysis is done in two ways: 1) through a simulated environment; 2) using the US yields data;

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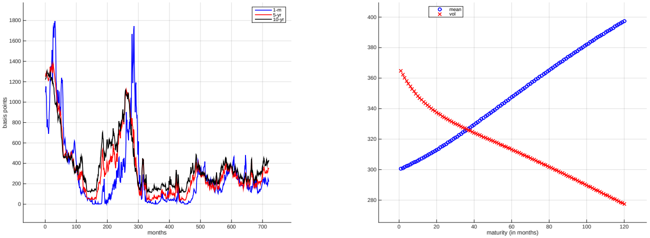
- Main advantage of a simulation environment:
  - We do know the **true forecasts**

6. Empirical Illustrations – a Simulated Environment



(a) Simulated yields

(b) Average Means and Vols of Simulated Yields

Figure: Statistics of simulated yields. Sample #1

6. Empirical Illustrations – a Simulated Environment

h	mat	Full-Sample $Gaussian_{NA}$	$Black_{NA}$	$Black_{ND}$	SV-$Black_{ND}$
3-m	1-m	10.8	0.91	0.96	0.75*
	2-yr	8.0	0.96	0.93	0.81*
	5-yr	7.9	1.00	0.90	0.82*
	10-yr	7.8	1.00	0.97	0.75*
1-yr	1-m	27.5	0.94	0.92	0.82*
	2-yr	26.1	0.97	0.91	0.77*
	5-yr	27.8	1.00	0.91	0.72*
	10-yr	26.9	1.01	0.96	0.75*

Table: **Bond Yield Forecast Errors** The symbol * indicates the best performance for each forecast horizon h and yield maturity mat.

6. Empirical Illustrations – a Simulated Environment

		ZLB Sample			
h	mat	$Gaussian_{NA}$	$Black_{NA}$	$Black_{ND}$	SV-$Black_{ND}$
3-m	1-m	7.8	0.69	0.58*	0.69
	2-yr	7.5	0.76	0.49*	0.61
	5-yr	7.2	1.01	0.74	0.62*
	10-yr	6.8	0.91	0.91	0.65*
1-yr	1-m	24.0	0.67	0.47*	0.74
	2-yr	23.4	0.87	0.66	0.54*
	5-yr	24.1	0.97	0.81	0.61*
	10-yr	21.3	1.00	1.04	0.70*

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6. Empirical Illustrations – a Simulated Environment

h	mat	Full-Sample G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}	ZLB Sample G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}
3-m	1-m	54.3	0.30	0.51	0.29*	68.9	0.15	0.14*	0.15
	2-yr	30.9	0.50	0.63	0.34*	36.8	0.34	0.18*	0.20
	5-yr	19.4	0.76	0.74	0.48*	25.2	0.42	0.42	0.29*
	10-yr	15.0	0.62	0.94	0.59*	20.1	0.28*	0.76	0.45
1-yr	1-m	63.6	0.72	0.53	0.42*	67.0	0.67	0.34*	0.38
	2-yr	39.9	1.04	0.61	0.46*	39.8	1.07	0.39*	0.40
	5-yr	27.9	1.20	0.63	0.58*	29.8	1.16	0.49	0.44*
	10-yr	18.6	1.23	0.86	0.76*	22.3	0.93	0.69	0.56*

Table: **Bond Yield Volatility Forecast Errors** The symbol * indicates the best performance.

6. Empirical Illustrations – a Simulated Environment

h	mat	Full-Sample				ZLB Sample			
		G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}	G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}
3-m	6-m	0.2	1.62	1.27	0.84*	0.3	0.96	0.60*	0.73
	2-yr	0.5	1.59	0.98	0.75*	0.6	1.06	0.51	0.51*
	5-yr	0.8	1.42	0.93	0.72*	1.0	1.13	0.68	0.46*
	9-yr	1.2	1.24	0.93	0.72*	1.3	1.01	0.82	0.61*
1-yr	6-m	0.2	1.59	1.19	0.92*	0.2	1.13	0.74*	0.82
	2-yr	0.3	1.57	1.01	0.79*	0.4	1.34	0.71	0.54*
	5-yr	0.6	1.42	0.95	0.68*	0.6	1.27	0.75	0.51*
	9-yr	0.9	1.24	0.98	0.67*	0.9	1.12	0.94	0.60*

Table: **Bond Sharpe Ratio Forecast Errors** The symbol * indicates the best performance.

6. Empirical Illustrations – a Simulated Environment

		Full-Sample				ZLB Sample			
h	mat	G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}	G_{NA}	B_{NA}	B_{ND}	SV-B_{ND}
	6m-9y	0.5	1.63	1.01	0.77*	0.7	1.04	0.49	0.45*
3-m	6m-4y	0.8	1.46	0.98	0.74*	1.0	1.10	0.73	0.49*
	5y-9y	1.0	1.33	0.93	0.71*	1.2	1.04	0.76	0.52*
	6m-9y	0.4	1.61	1.01	0.76*	0.4	1.39	0.72	0.52*
1-yr	6m-4y	0.7	1.42	0.99	0.67*	0.7	1.24	0.79	0.53*
	5y-9y	0.8	1.31	0.97	0.64*	0.8	1.27	0.95	0.58*

Table: **Bond Portfolio Sharpe Ratio Forecast Errors** The symbol * indicates the best performance.

6. Empirical Illustrations – historical U.S. data

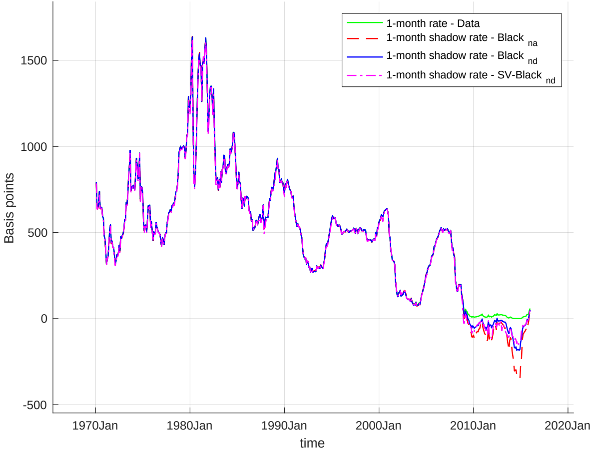

Figure: U.S. 1-month rate and 1-month shadow rate over sample period 1970:Jan - 2015:Dec

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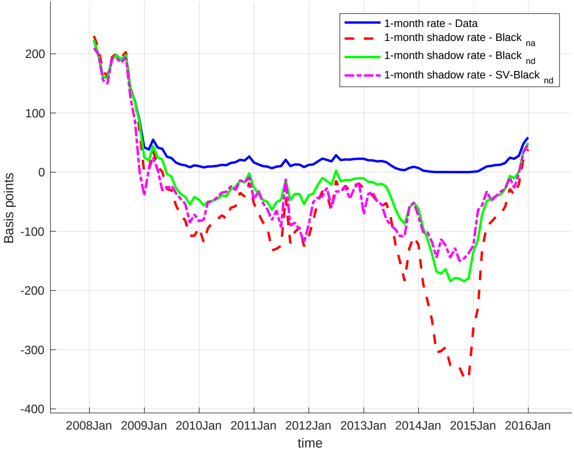

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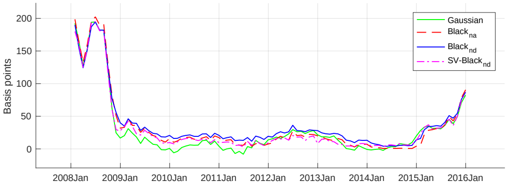
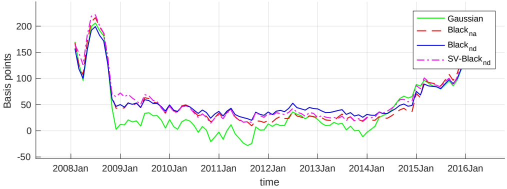


(a) 3-month forecast horizon



(b) 12-month forecast horizon

Figure: Forecasts of 1-m yields over sample period 2008:Jan - 2015:Dec

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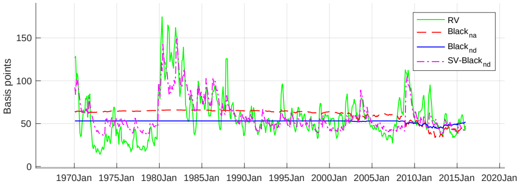

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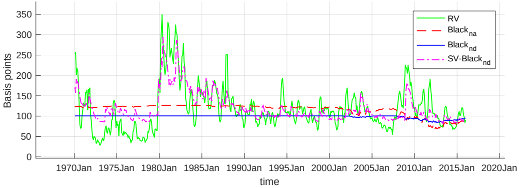

(b) 12-month forecast horizon

Figure: Volatility forecasts of 120-m yields over sample period 1970:Jan - 2015:Dec

6. Empirical Illustrations – historical U.S. data

		Entire Sample			ZLB Sample		
		B_{NA}	B_{ND}	SV-B_{ND}	B_{NA}	B_{ND}	SV-B_{ND}
3-m	Average	25.02	0.75	0.58^{*}	11.61	1.19	0.64^{*}
12-m	Average	45.81	0.81	0.67^{*}	28.84	1.04	0.78^{*}

Table: RMSE (in basis points) btw RV and volatility forecasts

6. Empirical Illustrations – historical U.S. data

		Entire Sample			ZLB Sample		
		B_{NA}	B_{ND}	SV-B_{ND}	B_{NA}	B_{ND}	SV-B_{ND}
3-m	Average	0.26	0.25	0.79*	0.67	0.62	0.80*
12-m	Average	0.25	0.24	0.79*	0.59	0.58	0.80*

Table: Correlation between volatility forecasts and RV

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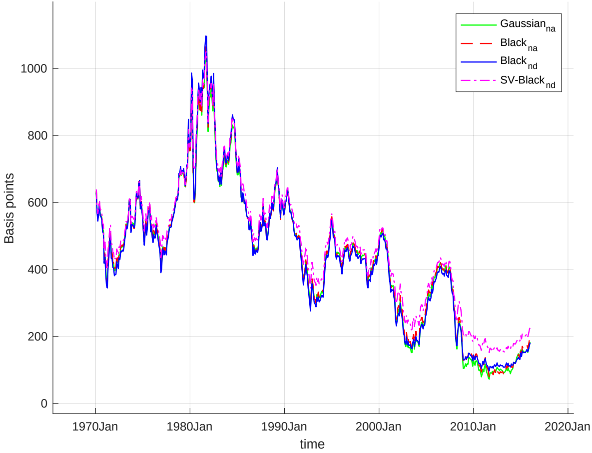

Figure: EH component of 120-m yields over sample period 1970:Jan - 2015:Dec

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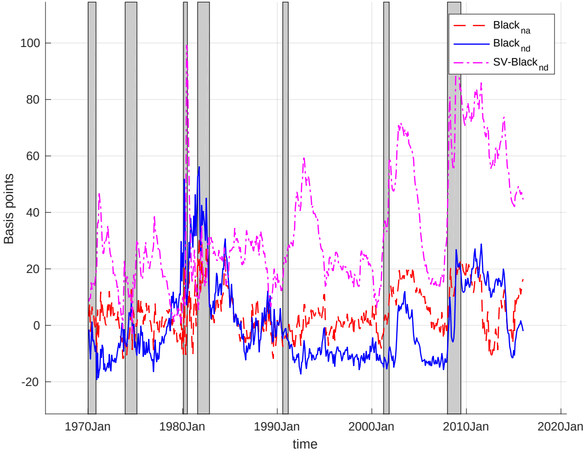

Figure: EH component of 120-m yields, relative to Gaussian model, over sample period 1970:Jan - 2015:Dec

Conclusion

- Propose an alternative approach to designing new tractable term structure models:
  1. We specify bond prices directly without going through any sdf.
  2. We then verify that these prices are free of dominant trading strategies.

- Imposition of lower bounds on yields is straightforward yet maintaining tractability and ease of implementation

- Simulation exercises show that our models can be at least comparable to some existing popular ZLB models