Shadow Interest Rate

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ZLB: monetary policy

Before ZLB
- Central banks lower policy rates to stimulate aggregate demand
- Economists rely on them to study monetary policy

Policy rates at ZLB
- Japan, US, Europe
- Unconventional policy tools
  - large-scale asset purchases (QE)
  - lending facilities
  - forward guidance
- Negative interest rates
ZLB: economic models

Term structure models

- Benchmark models
  - Gaussian ATSM allows undesirably negative interest rates
  - It is especially problematic when interest rates are low

- Our papers:
  - Wu and Xia (JMCB 2016): model US yield curve with ZLB
  - Wu and Xia (2017): model recent negative interest rates in Europe
ZLB: economic models

Term structure models

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New Keynesian models

- Benchmark models: no unconventional monetary policy
  - ZLB introduces structural break
  - Counterfactual economic implications
  - Computationally demanding
- Wu and Zhang (2016): incorporate unconventional monetary policy
  - Sensible economic implications
  - Tractable
Common theme: shadow rate

Black (1995)

\[ r_t = \max(s_t, r) \]
Contributions - Wu and Xia (JMCB 2016)

Measuring the Macroeconomic Impact of Monetary Policy at the ZLB

- Develop an analytical approximation for SRTSM
- Shadow rate has similar dynamic correlations with macro variables as the fed funds rate did previously
- Our shadow rates for US, Euro area, and UK are available at
  - Atlanta Fed
  - Haver Analytics
  - Thomson Reuters
  - Bloomberg
- Wu-Xia shadow rate has been discussed by
Contributions - Wu and Zhang (2016)

A Shadow Rate New Keynesian Model

- Present new empirical evidence relating the shadow rate to
  - private interest rates
  - Fed’s balance sheet
  - Taylor rule
- Propose a New Keynesian model with the shadow rate
  - accommodates both conventional and unconventional policies
- Microfoundations
  - QE
  - lending facilities
- Economic implications
  - a negative supply shock decreases output
    - data-consistent
    - contradicts standard models
  - government-spending multiplier is not larger than normal
Outlines

1. Wu-Xia shadow rate
2. Motivating shadow rate New Keynesian model
3. Microfoundations
4. Economic implications
Shadow rate

Black (1995):

\[ r_t = \max(s_t, r) \]

The shadow rate is affine

\[ s_t = \delta_0 + \delta_1 X_t \]

- \( r = 0.25 \), interest on reserves
- \( X_t \): 3 factors
Bond pricing

Factor dynamics:

\[ X_{t+1} = \mu + \rho X_t + \sum \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I). \]

Pricing kernel

\[ m_{t+1} = r_t + \frac{1}{2} \lambda_t' \lambda_t + \lambda_t' \varepsilon_{t+1} \]

where \( \lambda_t = \lambda_0 + \lambda_1 X_t \)

Pricing equation I

\[ P_{nt} = \mathbb{E}_t[\exp(-m_{t+1})P_{n-1,t+1}] \]

Pricing equation II

\[ P_{nt} = \mathbb{E}_t^Q[\exp(-r_t)P_{n-1,t+1}] \]
Bond pricing

Factor dynamics under risk-neutral measure $\mathbb{Q}$:

$$X_{t+1} = \mu^Q + \rho^Q X_t + \sum \varepsilon^Q_{t+1}, \quad \varepsilon^Q_{t+1} \sim \mathcal{N}(0, I).$$

where $\mu^Q = \mu - \Sigma \lambda_0$, and $\rho^Q = \rho - \Sigma \lambda_1$

Yield

$$y_{nt} = -\frac{1}{n} \log(P_{nt})$$

Forward rate from $t + n$ to $t + n + 1$

$$f_{nt} = (n + 1)y_{n+1,t} - ny_{nt}$$
Forward rates

Our approximation

\[ f_{nt} = r + \sigma_n^Q g \left( \frac{a_n + b'_n X_t - r}{\sigma_n} \right) \]

where \( g(z) = z \Phi(z) + \phi(z) \).

\[
\begin{align*}
  a_n &= \delta_0 + \delta'_1 \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right) \mu^Q - \frac{1}{2} \delta'_1 \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right) \Sigma' \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right)' \delta_1 \\
  b'_n &= \delta'_1 (\rho^Q)^n \\
  (\sigma_n^Q)^2 &\equiv \mathbb{V}_t^Q(s_{t+n}) = \sum_{j=0}^{n-1} \delta'_1 (\rho^Q)^j \Sigma' (\rho^Q)' \delta_1
\end{align*}
\]
Comparison to GATSM

**SRTSM**

\[
\begin{align*}
    s_t &= \delta_0 + \delta_1 X_t \\
    r_t &= \max(s_t, r)
\end{align*}
\]

Forward rate

\[
f_{nt} = r + \sigma_n^Q g \left( \frac{a_n + b_n' X_t - r}{\sigma_n^Q} \right)
\]

**GATSM**

\[
\begin{align*}
    r_t &= \delta_0 + \delta_1' X_t \\
    \text{Forward rate} \\
    f_{nt} &= a_n + b_n' X_t
\end{align*}
\]
Property of $g(.)$

\[ y = g(z) \]

\[ y = z \]

\[ \begin{align*}
    f_{nt}^{SR} &\approx r, \text{ at the ZLB} \\
    &\approx a_n + b'_n X_t = f_{nt}^G, \text{ when interest rates are high}
\end{align*} \]
Extended Kalman filter

State equation

\[ X_{t+1} = \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \varepsilon_{t+1} \sim N(0, I) \]

Observation equation

\[ f_{nt}^o = r + \sigma_n^Q g \left( \frac{a_n + b' X_t - r}{\sigma_n^Q} \right) + \eta_{nt}, \eta_{nt} \sim N(0, \omega) \]
Model fit

Figure: Average forward curve in 2012

SRTSM

GATSM

Log likelihood values

- SRTSM: 850; GATSM: 750
## Approximation error

Average absolute approximation error between 1990M1 and 2013M1 (in basis points)

<table>
<thead>
<tr>
<th></th>
<th>3M</th>
<th>6M</th>
<th>1Y</th>
<th>2Y</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
</tr>
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<tr>
<td>forward rate error</td>
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<td>0.02</td>
<td>0.04</td>
<td>0.13</td>
<td>0.69</td>
<td>1.14</td>
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<td>forward rate level</td>
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<td>384</td>
<td>435</td>
<td>551</td>
<td>600</td>
<td>636</td>
</tr>
<tr>
<td>yield error</td>
<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
<td>0.04</td>
<td>0.24</td>
<td>0.42</td>
<td>0.78</td>
</tr>
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Outline

1. Wu-Xia shadow rate

2. Motivating shadow rate New Keynesian model
   - SR as summary of unconventional monetary policy
   - Linear model

3. Microfoundations

4. Economic implications
Evidence 1: taper tantrum

- May 22, 2013: Bernanke told Congress Fed may decrease the size of QE
- shift in shadow rate summarizes this effect
Evidence 2: shadow rate and Fed’s balance sheet

Correlation
- QE1 - QE3: -0.94
Evidence 3: structural break test in FAVAR

Replace the fed funds rate with $s_t$ in Bernanke, Boivin, and Eliaasz (2005)

$$
\begin{align*}
x_t^m &= \mu^x + \rho^{xx} X_{t-1}^m + u_t^x \\
&+ \mathbb{1}(t < \text{December 2007}) \rho_1^{xs} S_{t-1} + \mathbb{1}(\text{December 2007} \leq t \leq \text{June 2009}) \rho_2^{xs} S_{t-1} + \mathbb{1}(t > \text{June 2009}) \rho_3^{xs} S_{t-1}
\end{align*}
$$

- monthly VAR(13)
- $x_t^m$: 3 underlying macro factors

Null hypothesis

$$H_0 : \rho_1^{xs} = \rho_3^{xs}$$

Likelihood ratio test: $\chi^2(39)$

- $p = 0.29$ for $s_t$
- $p = 0.0007$ for EFFR

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Evidence 4: shadow rate Taylor rule

\[ s_t = \beta_0 + \beta_1 s_{t-1} + \beta_2 (y_t - y^n_t) + \beta_3 \pi_t + \varepsilon_t \]

Test for structural break

- \( F \) statistic = 2
- Critical value: 2.37
- No structural break
Evidence 5: shadow rate and private rates

- Private rates are the relevant rates for agents and the economy
- Correlations: 0.8
- \( r_B^t = s_t + rp \)
Summary

Shadow rate summarizes unconventional monetary policy

- Taper tantrum
- Fed’s balance sheet

There is no structural break in

- relationship between macro variables and the shadow rate
- shadow rate Taylor rule

Private rates

- are the relevant interest rates for economic agents
- respond to unconventional monetary policy
- reflect the overall effect of UMP on the economy
- the shadow rate is a sensible summary
Shadow rate New Keynesian model

Definition

The shadow rate New Keynesian model consists of the shadow rate IS curve

\[ y_t = -\frac{1}{\sigma} (s_t - \mathbb{E}_t \pi_{t+1} - s) + \mathbb{E}_t y_{t+1}, \]

New Keynesian Phillips curve

\[ \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa (y_t - y^n_t), \]

and shadow rate Taylor rule

\[ s_t = \phi_s s_{t-1} + (1 - \phi_s) \left[ \phi_y (y_t - y^n_t) + \phi_{\pi} \pi_t + s \right]. \]
Outline

1. Wu-Xia shadow rate

2. Motivating shadow rate New Keynesian model

3. Microfoundations
   - Microfoundations I: QE
   - Microfoundations II: lending facilities

4. Economic implications
Microfoundation for SRNKM I: QE

The risk premium channel

- government purchases outstanding loans
- decrease interest rates through reducing risk premium
  - Gagnon et al. (2011) and Hamilton and Wu (2012)
- the same mechanism works for government bonds or corporate bonds
Households’ problem

Households’ utility function

\[ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right) \]

budget constraint

\[ C_t + \frac{B_t^H}{P_t} = \frac{R_{t-1}^B B_{t-1}^H}{P_t} + W_t L_t + T_t \]

Euler equation

\[ C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left[ \frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right] \]

The linear Euler equation

\[ y_t = -\frac{1}{\sigma} \left( r_t^B - \mathbb{E}_t \pi_{t+1} - r^B \right) + \mathbb{E}_t y_{t+1}, \]
Bond return and policy rate

Define

\[ rp_t \equiv r^B_t - r_t \]

- interpreted as convenience yield by Krishnamurthy and Vissing-Jorgensen (2012)
- Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Hamilton and Wu (2012) suggest

\[ rp'_t(b^G_t) < 0 \Rightarrow rp_t(b^G_t) = rp - \varsigma (b^G_t - b^G) \]

During normal times

- \( b^G_t = b^G \)
- \( rp_t(b^G) = rp \Rightarrow r^B_t = r_t + rp \)

At the ZLB

- QE \(\rightarrow b^G_t \uparrow \rightarrow rp_t \downarrow \rightarrow r^B_t \downarrow \)
Shadow rate equivalence for QE

Euler equation

\[ y_t = -\frac{1}{\sigma} \left( r^B_t - \mathbb{E}_t \pi_{t+1} - r^B \right) + \mathbb{E}_t y_{t+1}, \]

- During normal times, \( b^G_t = b^G \), \( r_t = s_t \)

\[ r^B_t = r_t + r_p = s_t + r_p \]

- At the ZLB, \( r_t = 0 \)

\[ r^B_t = r_p t = r_p - \varsigma (b^G_t - b^G) = s_t + r_p \]

if \( s_t = -\varsigma (b^G_t - b^G) \)
Shadow rate equivalence for QE

Proposition

The shadow rate New Keynesian model represented by the shadow rate IS curve

\[ y_t = -\frac{1}{\sigma} (s_t - \mathbb{E}_t \pi_{t+1} - s) + \mathbb{E}_t y_{t+1}, \]

New Keynesian Phillips Curve, and shadow rate Taylor rule

\[ s_t = \phi_s s_{t-1} + (1 - \phi_s) [\phi_y (y_t - y^n_t) + \phi_\pi \pi_t + s] \]

nests both conventional Taylor interest rate rule and QE operation that changes risk premium if

\[
\begin{align*}
  r_t &= s_t, \quad b^G_t = b^G & \text{for } s_t \geq 0 \\
  r_t &= 0, \quad b^G_t = b^G - \frac{s_t}{\varsigma} & \text{for } s_t < 0.
\end{align*}
\]
Quantifying assumption in proposition

\[ s_t = -\varsigma (b_t^G - b^G) \]

- Linear assumption: correlation = 0.92
- \( \varsigma = 1.83 \)
  - Fed increases its bond holdings by 1%, the shadow rate decreases by 0.0183%  
  - QE1: 490 billion to 2 trillion \( \Rightarrow \) 2.5% decrease in the shadow rate  
  - QE3: 2.6 trillion to 4.2 trillion \( \Rightarrow \) 0.9% decrease in the shadow rate
Outline

1. Wu-Xia shadow rate

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   - Microfoundations I: QE
   - Microfoundations II: lending facilities

4. Economic implications
Microfoundation for SRNKM II: lending facilities

Lending facilities

- extend loans to the private sector ⇒ change the loan-to-value ratio
- Example: Term Asset-Backed Securities Loan Facility in the US

Combine this with a tax policy

- tax (subsidy) on the interest rate income (payment)
Model features

Entrepreneurs

- produce intermediate goods with labor and capital
- maximize utility
- discount factor $\gamma < \beta$
- borrow from households with a loan-to-value ratio $M$
- accumulate capital
- use capital as collateral

Government policy at the ZLB

- lending facilities
  - lend directly to entrepreneurs
  - change the loan-to-value ratio from $M$ to $M_t$
- tax (subsidy) on the interest rate income (payment)
Entrepreneurs’ problem

Utility function

\[
\max \mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E
\]

production function

\[
Y_t^E = AK_{t-1}^\alpha (L_t)^{1-\alpha}
\]

capital accumulation

\[
K_t = I_t + (1 - \delta)K_{t-1}
\]

budget constraint

\[
\frac{Y_t^E}{X_t} + \tilde{B}_t = \frac{R_{t-1}^B \tilde{B}_{t-1}}{T_{t-1} \Pi_t} + W_t L_t + I_t + C_t^E
\]

borrowing constraint

\[
\tilde{B}_t \leq M_t \mathbb{E}_t \left( K_t \Pi_{t+1} / R_t^B \right)
\]
Entrepreneurs’ FOCs

Labor demand

\[ W_t = \frac{(1 - \alpha)AK_{t-1}^{\alpha}L_t^{-\alpha}}{X_t} \]

Euler equation

\[ \frac{1}{C_t^E} \left( 1 - \frac{M_t^E}{R_t^B} \prod_{t+1} \right) = \gamma \mathbb{E}_t \left[ \frac{1}{C_{t+1}^E} \left( \frac{\alpha Y_{t+1}^E}{X_{t+1}K_{t+1}} - \frac{M_t}{T_t} + 1 - \delta \right) \right] \]
Households’ problem

Households’ utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right)$$

budget constraint

$$C_t + \tilde{B}_t^H = \frac{R_{t-1}^B \tilde{B}_{t-1}^H}{\bar{T}_{t-1} \Pi_t} + W_t L_t + T_t$$

Euler equation

$$C_{t-\sigma}^t = \beta E_t \left( R_t^B \frac{C_{t+1}^{t-\sigma}}{\Pi_{t+1} \bar{T}_t} \right)$$

labor supply

$$W_t = C_t^\sigma L_t^\eta$$
Sources of funding

Entrepreneurs’ borrowing constraint

\[ \tilde{B}_t \leq M_t \mathbb{E}_t \left( \frac{K_t \Pi_{t+1}}{R_t^B} \right) \]

Households lend

\[ \tilde{B}_t^H \leq M \mathbb{E}_t \left( \frac{K_t \Pi_{t+1}}{R_t^B} \right) \]

- During normal times \( \tilde{B}_t = \tilde{B}_t^H \), and \( M_t = M \)
- At the ZLB \( M_t > M \)

Government lends the rest

\[ \tilde{B}_t^G = (M_t - M) \mathbb{E}_t \left( \frac{K_t \Pi_{t+1}}{R_t^B} \right) \]
Conventional and unconventional policy

Suppose $R_t^B = R_t RP$

Conventional and unconventional policy tools appear in the model in pairs:

- $R_t/T_t$ – HH Euler equation, HH&E budget constraints
- $R_t/M_t$ – E borrowing constraint, E Euler equation
- $M_t/T_t$ – E Euler equation

Decreasing $R_t$ is equivalent to increasing $T_t$ and $M_t$. 
Shadow rate equivalence for lending facilities

Proposition

If

\[
\begin{align*}
R_t &= S_t, \quad T_t = 1, \quad M_t = M \quad \text{for } S_t \geq 1 \\
T_t &= M_t / M = 1 / S_t \quad \text{for } S_t < 1,
\end{align*}
\]

then \( R_t / T_t = S_t \), \( R_t / M_t = S_t / M \), \( M_t / T_t = M \) \( \forall S_t \).

- \( S_t \) summarizes both conventional and unconventional policies

- Equivalence in the non-linear model
Shadow rate equivalence for lending facilities

Proposition

The shadow rate New Keynesian model represented by the Euler equation

\[ c_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t \pi_{t+1} - s) + \mathbb{E}_t c_{t+1} \]

the shadow rate Taylor rule, Phillips curve, ..., nests both conventional Taylor interest rate rule and lending facility – tax policy if

\[
\begin{cases}
    r_t = s_t, \tau_t = 0, m_t = m & \text{for } s_t \geq 0 \\
    \tau_t = m_t - m = -s_t & \text{for } s_t < 0.
\end{cases}
\]
1. Wu-Xia shadow rate

2. Motivating shadow rate New Keynesian model

3. Microfoundations

4. Economic implications
   - Economic implication I: negative supply shock
   - Economic implication II: government spending multiplier
Economic implication I: negative supply shock

Technology shock

\[ a_t ↓ = \rho a_{t-1} + e_{a,t} ↓ \]

Phillips Curve

\[ \pi_t ↑ = \beta E_t \pi_{t+1} + \kappa(y_t - \bar{y}) - \frac{\kappa(1 + \eta)}{\sigma + \eta} a_t ↓ \]
Economic implication I: negative supply shock

Standard model

Monetary policy

\[ r_t = \max\{\phi_s s_{t-1} + (1 - \phi_s) \left[ \phi_y (y_t - y_t^n) + \phi_\pi \pi_t + s \right], 0\} \]

Real interest rate

\[ rr_t = r_t - \mathbb{E}_t [\pi_{t+1}] \]

IS curve

\[ y_t = -\frac{1}{\sigma} (rr_t - r) + \mathbb{E}_t y_{t+1} \]

normal times: \( \pi \uparrow \rightarrow r \uparrow \uparrow \rightarrow rr \uparrow \rightarrow y \downarrow \)

ZLB without UMP: \( \pi \uparrow \rightarrow r = 0 \rightarrow rr \downarrow \rightarrow y \uparrow \) Counterfactual
Economic implication I: negative supply shock

Shadow rate model

Shadow rate Taylor rule

\[ s_t = \phi_s s_{t-1} + (1 - \phi_s) \left[ \phi_y (y_t - y_t^n) + \phi_\pi \pi_t + s \right] \]

Real interest rate

\[ rr_t = s_t - \mathbb{E}_t [\pi_{t+1}] \]

Shadow rate IS curve

\[ y_t = -\frac{1}{\sigma} (s_t - \mathbb{E}_t \pi_{t+1} - r) + \mathbb{E}_t y_{t+1} \]

normal times: \[ \pi \uparrow \rightarrow r \uparrow \uparrow \rightarrow rr \uparrow \rightarrow y \downarrow \]

ZLB without UMP: \[ \pi \uparrow \rightarrow r = 0 \rightarrow rr \downarrow \rightarrow y \uparrow \text{Counterfactual} \]

ZLB with UMP: \[ \pi \uparrow \rightarrow s \uparrow \uparrow \rightarrow rr \uparrow \rightarrow y \downarrow \text{Data consistent} \]
Extended model

Iacoviello (2005)

- Patient households
  - consume, work, hold house, lend
- Impatient households
  - consume, work, hold house
  - borrow with collateral
- Entrepreneurs
  - consume, invest, hold houses
  - borrow with collateral
  - produce intermediate goods using housing, capital, and labor
- Retailers
  - monopolistically competitive
  - Calvo sticky
- Government
  - Taylor rule
Difference from Iacoviello (2005)

- Unconventional policy
  - QE: time-varying risk premium
  - lending facilities: time-varying loan-to-value ratio
  - time-varying tax
- government spending
- preference shock: create ZLB
Models and methodology

Notation

- **Standard model**: without unconventional policy: \( r_t = 0 \)
- **Shadow rate model**: with unconventional policy: \( s_t < 0 \)

Methodology for shadow rate model

- solve linear model with shadow rate
- then use propositions mapping into various UMP
Preference shock and the ZLB

Preference shock

% dev. from S.S.

0 0.05 0.1 0.15 0.2 0.25

5 10 15 20 25 30 35 40

r_t policy rate

percentage

0 0.5 1 1.5 2 2.5

5 10 15 20 25 30 35 40

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Economic implication I: negative supply shock

Note: **blue**: shadow rate NK model with UMP; **red**: standard NK model without UMP
Economic implication I: negative supply shock

π_t inflation

r_t policy rate

s_t shadow rate

r_t^{B-\tau} private rate

r_t real interest rate

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
Economic implication I: negative supply shock

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
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Economic implication II: government spending multiplier

Government spending shock

\[ g_t \uparrow = (1 - \rho_g)g + \rho g g_{t-1} + e_{g,t} \uparrow \]

Market-clearing condition

\[ y_t \uparrow = c_y c_t + g_y g_t \uparrow \]

Phillips Curve

\[ \pi_t \uparrow = \beta E_t \pi_{t+1} + \frac{\kappa}{\delta + \eta} \left( \sigma (c_t - c) + \eta (y_t \uparrow - y) \right) \]
Economic implication II: government spending multiplier

**Standard model**

Monetary policy

\[ r_t = \max \{ \phi_s s_{t-1} + (1 - \phi_s) [\phi_y (y_t - y_t^n) + \phi_\pi \pi_t + s], 0 \} \]

Real interest rate

\[ rr_t = r_t - \mathbb{E}_t[\pi_{t+1}] \]

IS curve

\[ c_t = -\frac{1}{\sigma} (rr_t - r) + \mathbb{E}_t c_{t+1} \]

**Normal times:** \( \pi \uparrow y \uparrow \rightarrow r \uparrow \rightarrow rr \uparrow \rightarrow c \downarrow \rightarrow \Delta y < \Delta g \)

**ZLB without UMP:** \( \pi \uparrow y \uparrow \rightarrow r = 0 \rightarrow rr \downarrow \rightarrow c \uparrow \rightarrow \Delta y > \Delta g \)
Economic implication II: government spending multiplier

Shadow rate model

Shadow rate Taylor rule

\[ s_t = \phi_s s_{t-1} + (1 - \phi_s) [\phi_y (y_t - y_t^n) + \phi_\pi \pi_t + s] \]

Real interest rate

\[ rr_t = s_t - \mathbb{E}_t [\pi_{t+1}] \]

Shadow rate IS curve

\[ c_t = -\frac{1}{\sigma} (s_t - \mathbb{E}_t \pi_{t+1} - r) + \mathbb{E}_t c_{t+1} \]

normal times: \[ \pi \uparrow \ y \uparrow \ \rightarrow \ r \uparrow \uparrow \ \rightarrow \ rr \uparrow \ \rightarrow \ c \downarrow \ \rightarrow \ \Delta y < \Delta g \]

ZLB without UMP: \[ \pi \uparrow \ y \uparrow \ \rightarrow \ r = 0 \ \rightarrow \ rr \downarrow \ \rightarrow \ c \uparrow \ \rightarrow \ \Delta y > \Delta g \]

ZLB with UMP: \[ \pi \uparrow \ y \uparrow \ \rightarrow \ s \uparrow \uparrow \ \rightarrow \ rr \uparrow \ \rightarrow \ c \downarrow \ \rightarrow \ \Delta y < \Delta g \]
Economic implication II: government spending multiplier

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
Economic implication II: government spending multiplier

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
Economic implication II: government spending multiplier

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
Economic implication II: government spending multiplier

Note: blue: shadow rate NK model with UMP; red: standard NK model without UMP
Computational advantages

The ZLB imposes one of the biggest challenges for solving and estimating these models.

Existing methods
- have undesirable economic implications
- computationally demanding

Shadow rate model
- no structural break
- can rely on traditional methods
- unique solution
Conclusion

Wu and Xia (JMCB 2016)
- provide an analytical approximation for SRTSM
- Wu-Xia shadow rate summarizes unconventional monetary policy

Wu and Zhang (2016)
- propose a New Keynesian model with the shadow rate
- map unconventional policy tools into the shadow rate framework
- make counterfactual results in standard NK model disappear
- computationally tractable
Related work: Wu and Xia (2017)

Time-Varying Lower Bound of Interest Rates in Europe

- A new shadow rate term structure model with $r_t = \max(s_t, r_t)$
  - $r_t = r_t^d + s_p t$
    - Discrete policy lower bound $r_t^d$
    - Non-constant spread $s_p t$
  - Agents are forward looking
  - Derive bond prices
- Asymmetry
  - 10 bp drop in the ELB decreases the 10-year rate by 6.5-8.5 bp
  - 10 bp initial rise in the ELB increases the 10-year rate by 9-14 bp
Gürkaynak, Sack, and Wright (2007) dataset
  ▶ monthly, January 1990 - December 2013
  ▶ maturities: 3 and 6 months, 1, 2, 5, 7 and 10 years

Normalization with repeated eigenvalues

\[ \rho_Q = \begin{bmatrix} \rho_1^Q & 0 & 0 \\ 0 & \rho_2^Q & 1 \\ 0 & 0 & \rho_2^Q \end{bmatrix}. \]
FAVAR

Replace the fed funds rate with $s_t$ in Bernanke, Boivin, and Eliasz (2005)

$$Y_t^m = a_m + b_x x_t^m + b_s s_t + \eta_t^m, \quad \eta_t^m \sim N(0, \Omega)$$

- $Y_t^m$: 97 economic variables from 1960 to 2013
- $x_t^m$: 3 underlying macro factors

Factor dynamics:

$$\begin{bmatrix} x_t^m \\ s_t \end{bmatrix} = \begin{bmatrix} \mu^x \\ \mu^s \end{bmatrix} + \begin{bmatrix} \rho^{xx} & \rho^{xs} \\ \rho^{sx} & \rho^{ss} \end{bmatrix} \begin{bmatrix} X_{t-1}^m \\ S_{t-1} \end{bmatrix} + \Sigma^m \begin{bmatrix} \varepsilon_t^m \\ \varepsilon_{MP}^t \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_t^m \\ \varepsilon_{MP}^t \end{bmatrix} \sim N(0, I)$$

- monthly VAR(13)
- $\Sigma^m$: Cholesky decomposition
## Rubustness

<table>
<thead>
<tr>
<th></th>
<th>$p$-value for $\rho_1^{xs} = \rho_3^{xs}$</th>
<th>$p$-value for $\rho_1^{sx} = \rho_3^{sx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>0.29</td>
<td>1.00</td>
</tr>
<tr>
<td>A1 estimate $r$</td>
<td>0.18</td>
<td>1.00</td>
</tr>
<tr>
<td>A2 2-factor SRTSM</td>
<td>0.13</td>
<td>0.97</td>
</tr>
<tr>
<td>A3 Fama-Bliss</td>
<td>0.38</td>
<td>1.00</td>
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<tr>
<td>A4 5-factor FAVAR</td>
<td>0.70</td>
<td>1.00</td>
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<tr>
<td>A5 6-lag FAVAR</td>
<td>0.09</td>
<td>0.98</td>
</tr>
<tr>
<td>7-lag FAVAR</td>
<td>0.19</td>
<td>0.97</td>
</tr>
<tr>
<td>12-lag FAVAR</td>
<td>0.22</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Lending facilities

\[
\begin{align*}
\text{ct} & = - \frac{1}{\sigma}(r_t^B - \tau_t - E_t \pi_{t+1} - r - rp) + E_t c_{t+1} \\
\Rightarrow \text{ct} & = - \frac{1}{\sigma}(s_t - E_t \pi_{t+1} - r) + E_t c_{t+1}
\end{align*}
\]

\[
\begin{align*}
C^E c_t^E & = \alpha \frac{Y}{X} (y_t - x_t) + Bb_t - R^B B(r_t^{B-1} + b_{t-1} - \tau_{t-1} - \pi_{t-1}) - li_t + \Lambda_1 \\
\Rightarrow C^E c_t^E & = \alpha \frac{Y}{X} (y_t - x_t) + Bb_t - R^B B(s_{t-1} + rp + b_{t-1} - \pi_{t-1}) - li_t + \Lambda_1
\end{align*}
\]

\[
\begin{align*}
b_t & = E_t (k_t + \pi_{t+1} + m_t - r_t^B) \\
\Rightarrow b_t & = E_t (k_t + \pi_{t+1} + m - s_t - rp)
\end{align*}
\]
Lending facilities

\[ 0 = \left(1 - \frac{M}{RB}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma \alpha Y}{XK} \mathbb{E}_t (y_{t+1} - x_{t+1} - k_t) + \frac{M}{RB} \mathbb{E}_t (\pi_{t+1} - r^B_t + m_t) + \gamma M(\tau_t - m_t) + \Lambda_2 \]

\[ \Rightarrow 0 = \left(1 - \frac{M}{RB}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma \alpha Y}{XK} \mathbb{E}_t (y_{t+1} - x_{t+1} - k_t) + \frac{M}{RB} \mathbb{E}_t (\pi_{t+1} - s_t - rp + m) - \gamma Mm + \Lambda_2 \]
## Calibration

<table>
<thead>
<tr>
<th>para</th>
<th>description</th>
<th>source</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>discount factor of patient households</td>
<td>Iacoviello (2005)</td>
<td>0.99</td>
</tr>
<tr>
<td>$\beta^I$</td>
<td>discount factor of impatient households</td>
<td>Iacoviello (2005)</td>
<td>0.95</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>discount factor of entrepreneurs</td>
<td>Iacoviello (2005)</td>
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</tr>
<tr>
<td>$j$</td>
<td>steady-state weight on housing services</td>
<td>Iacoviello (2005)</td>
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<tr>
<td>$\eta$</td>
<td>labor supply aversion</td>
<td>Iacoviello (2005)</td>
<td>0.01</td>
</tr>
<tr>
<td>$\mu$</td>
<td>capital share in production</td>
<td>Iacoviello (2005)</td>
<td>0.3</td>
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<tr>
<td>$\nu$</td>
<td>housing share in production</td>
<td>Iacoviello (2005)</td>
<td>0.03</td>
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<tr>
<td>$\delta$</td>
<td>capital depreciation rate</td>
<td>Iacoviello (2005)</td>
<td>0.03</td>
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<tr>
<td>$X$</td>
<td>steady state gross markup</td>
<td>Iacoviello (2005)</td>
<td>1.05</td>
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<tr>
<td>$\theta$</td>
<td>probability that cannot re-optimize</td>
<td>Iacoviello (2005)</td>
<td>0.75</td>
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<tr>
<td>$\alpha$</td>
<td>patient households' wage share</td>
<td>Iacoviello (2005)</td>
<td>0.64</td>
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<tr>
<td>$M^E$</td>
<td>loan-to-value ratio for entrepreneurs</td>
<td>Iacoviello (2005)</td>
<td>0.89</td>
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<tr>
<td>$M^I$</td>
<td>loan-to-value ratio for impatient households</td>
<td>Iacoviello (2005)</td>
<td>0.55</td>
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<tr>
<td>$r_R$</td>
<td>interest rate persistence</td>
<td>Iacoviello (2005)</td>
<td>0.73</td>
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<tr>
<td>$r_Y$</td>
<td>interest rate response to output</td>
<td>Iacoviello (2005)</td>
<td>0.27</td>
</tr>
<tr>
<td>$r_I$</td>
<td>interest rate response to inflation</td>
<td>Iacoviello (2005)</td>
<td>0.13</td>
</tr>
<tr>
<td>$\xi^G$</td>
<td>steady-state government-spending-to-output ratio</td>
<td>Fernandez-Villaverde et al. (2015)</td>
<td>0.20</td>
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<tr>
<td>$\rho_a$</td>
<td>autocorrelation of technology shock</td>
<td>Fernandez-Villaverde et al. (2015)</td>
<td>0.90</td>
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<td>Fernandez-Villaverde et al. (2015)</td>
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<td>autocorrelation of discount rate shock</td>
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<td>$\sigma_a$</td>
<td>standard deviation of technology shock</td>
<td>Fernandez-Villaverde et al. (2015)</td>
<td>0.0025</td>
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<td>$\sigma_g$</td>
<td>standard deviation of government-spending shock</td>
<td>Fernandez-Villaverde et al. (2015)</td>
<td>0.0025</td>
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<tr>
<td>$\sigma_\beta$</td>
<td>standard deviation of discount rate shock</td>
<td>Fernandez-Villaverde et al. (2015)</td>
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<tr>
<td>$\xi_p$</td>
<td>price indexation</td>
<td>Smets and Wouters (2007)</td>
<td>0.24</td>
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<tr>
<td>$\Pi$</td>
<td>steady-state inflation</td>
<td>2% annual inflation</td>
<td>1.005</td>
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<tr>
<td>$B^G$</td>
<td>steady-state government bond holdings</td>
<td>no gov. intervention in private bond market</td>
<td>0</td>
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<tr>
<td>$T$</td>
<td>steady-state tax/subsidy on interest rate income/payment</td>
<td>no tax in normal times</td>
<td>1</td>
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<tr>
<td>$rp$</td>
<td>steady-state risk premium</td>
<td>3.6% risk premium annually</td>
<td>1.009</td>
</tr>
</tbody>
</table>
Preference shock and the ZLB

1. $s_t$ shadow rate
2. $r_t$ policy rate
3. $i_t^B - i_t$ private rate
4. $\pi_t$ inflation
5. $r_t$ real interest rate
6. $\rho_t$ risk premium - QE
7. $\tau_t$ tax - Facilities
8. $\Delta_t$ loan-to-value ratio - Facilities
9. $Y_t$ output
10. $C_t$ consumption
11. $I_t$ investment
12. Preference shock

ZLB period
without UMP
with UMP