Structural Interpretation of Vector Autoregressions with Incomplete Identification: Revisiting the Role of Oil Supply and Demand Shocks

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- Causal interpretation of correlations requires information about economic structure
- Standard approaches can be given a Bayesian interpretation (identifying assumptions as Bayesian prior)
- Formal Bayesian methods permit less dogmatic and more flexible structural inference

Outline

I. Bayesian inference for structural vector autoregressions
   (Baumeister and Hamilton, Econometrica, 2015)

II. A Bayesian interpretation of traditional identification
- Example 1: Kilian (AER, 2009)
- Example 2: Kilian and Murphy (JEEA, 2012)

III. Full Bayesian treatment of the role of oil supply and demand shocks
   A. Inventories and measurement error
   B. Specification of prior
   C. Treatment of older data

IV. Empirical results
   A. Baseline results
   B. Sensitivity analysis

I. Bayesian inference for structural vector autoregressions

Structural model of interest:

$$\underset{(n\times n)}{\mathbf{A}}\underset{(n\times 1)}{\mathbf{y}_t} = \boldsymbol{\lambda} + \mathbf{B}_1\mathbf{y}_{t-1} + \cdots + \mathbf{B}_m\mathbf{y}_{t-m} + \mathbf{u}_t$$

$\mathbf{u}_t \sim$ i.i.d. $N(\mathbf{0}, \mathbf{D})$

$\mathbf{D}$ diagonal

Example: supply and demand

$$q_t = k^s + \alpha^s p_t + b^s_{11} p_{t-1} + b^s_{12} q_{t-1} + b^s_{21} p_{t-2} + b^s_{22} q_{t-2} + \cdots + b^s_{m1} p_{t-m} + b^s_{m2} q_{t-m} + u^s_t$$

$$q_t = k^d + \beta^d p_t + b^d_{11} p_{t-1} + b^d_{12} q_{t-1} + b^d_{21} p_{t-2} + b^d_{22} q_{t-2} + \cdots + b^d_{m1} p_{t-m} + b^d_{m2} q_{t-m} + u^d_t$$

$$\mathbf{A} = \begin{bmatrix} 1 & -\alpha^s \\ 1 & -\beta^d \end{bmatrix}$$

Supply and demand example:

4 structural parameters in $\mathbf{A}, \mathbf{D}$

$(\alpha^s, \beta^d, d_{11}, d_{22})$

only 3 parameters known from $\mathbf{\Omega}$

$(\omega_{11}, \omega_{12}, \omega_{22})$

We can achieve full identification if we know the value of α^s (e.g., $\alpha^s = 0$ gives Cholesky).

We can achieve partial identification from

$\alpha^s \geq 0, \; \beta^d \leq 0.$

Consider Bayesian approach where we begin with arbitrary prior $p(\mathbf{A})$

E.g., prior beliefs about supply and demand elasticities in the forms of densities $p(\alpha^s, \beta^d)$

$$\mathbf{A} = \begin{bmatrix} 1 & -\alpha^s \\ 1 & -\beta^d \end{bmatrix}$$

Bayesian begins with prior beliefs before seeing data:

$$p(\mathbf{A}, \mathbf{D}, \mathbf{B}) = p(\mathbf{A})p(\mathbf{D}|\mathbf{A})p(\mathbf{B}|\mathbf{D}, \mathbf{A})$$

Goal is to calculate posterior beliefs after seeing data $\mathbf{Y}_T = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_T\}$

$$p(\mathbf{A}, \mathbf{D}, \mathbf{B}|\mathbf{Y}_T) = p(\mathbf{A}|\mathbf{Y}_T)p(\mathbf{D}|\mathbf{A}, \mathbf{Y}_T)p(\mathbf{B}|\mathbf{D}, \mathbf{A}, \mathbf{Y}_T)$$

prior:

$$\mathbf{b}_i|\mathbf{A},\mathbf{D} \sim N(\mathbf{m}_i, d_{ii}\mathbf{M}_i)$$

posterior:

$$\mathbf{b}_i|\mathbf{A},\mathbf{D},\mathbf{Y}_T \sim N(\mathbf{m}_i^*, d_{ii}\mathbf{M}_i^*)$$

As $T \to \infty$, $\mathbf{m}_i^* \to$ true value and $\mathbf{M}_i^* \to \mathbf{0}$

prior:

$$d_{ii}^{-1}|\mathbf{A} \sim \Gamma(\kappa_i, \tau_i)$$

posterior:

$$d_{ii}^{-1}|\mathbf{A}, \mathbf{Y}_T \sim \Gamma(\kappa_i^*, \tau_i^*)$$

As $T \to \infty,\ d_{ii} \xrightarrow{p}$ true value

If identified: as $T \to \infty$,

$p(\mathbf{A}|\mathbf{Y}_T) \to$ Dirac delta at true $\mathbf{A}_0$

If unidentified:

$p(\mathbf{A}|\mathbf{Y}_T) \to$ prior $p(\mathbf{A})$ confined to identified set $\{\mathbf{A} : \mathbf{A}\boldsymbol{\Omega}_0\mathbf{A}'$ is diagonal$\}$

II. A Bayesian interpretation of traditional identification

q = quantity of oil produced

y = measure of economic activity

p = real price of oil

oil supply:

$$q_t = \alpha_{qy} y_t + \alpha_{qp} p_t + \mathbf{b}_1' \mathbf{x}_{t-1} + u_{1t}$$

economic activity:

$$y_t = \alpha_{yq} q_t + \alpha_{yp} p_t + \mathbf{b}_2' \mathbf{x}_{t-1} + u_{2t}$$

inverse of oil demand curve:

$$p_t = \alpha_{pq} q_t + \alpha_{py} y_t + \mathbf{b}_3' \mathbf{x}_{t-1} + u_{3t}$$

Note: α_{pq} = inverse of short-run price-elasticity of oil demand

$$\mathbf{A}\mathbf{y}_t = \boldsymbol{\lambda} + \mathbf{B}_1\mathbf{y}_{t-1} + \cdots + \mathbf{B}_m\mathbf{y}_{t-m} + \mathbf{u}_t$$

$$\mathbf{A} = \begin{bmatrix} 1 & -\alpha_{qy} & -\alpha_{qp} \\ -\alpha_{yq} & 1 & -\alpha_{yp} \\ -\alpha_{pq} & -\alpha_{py} & 1 \end{bmatrix}$$

Example 1: Kilian (AER 2009)
(Cholesky identification)

$$\alpha_{qy} = \alpha_{qp} = \alpha_{yp} = 0$$

oil supply:

$$q_t = \alpha_{qy} y_t + \alpha_{qp} p_t + \mathbf{b}_1' \mathbf{x}_{t-1} + u_{1t}$$

economic activity:

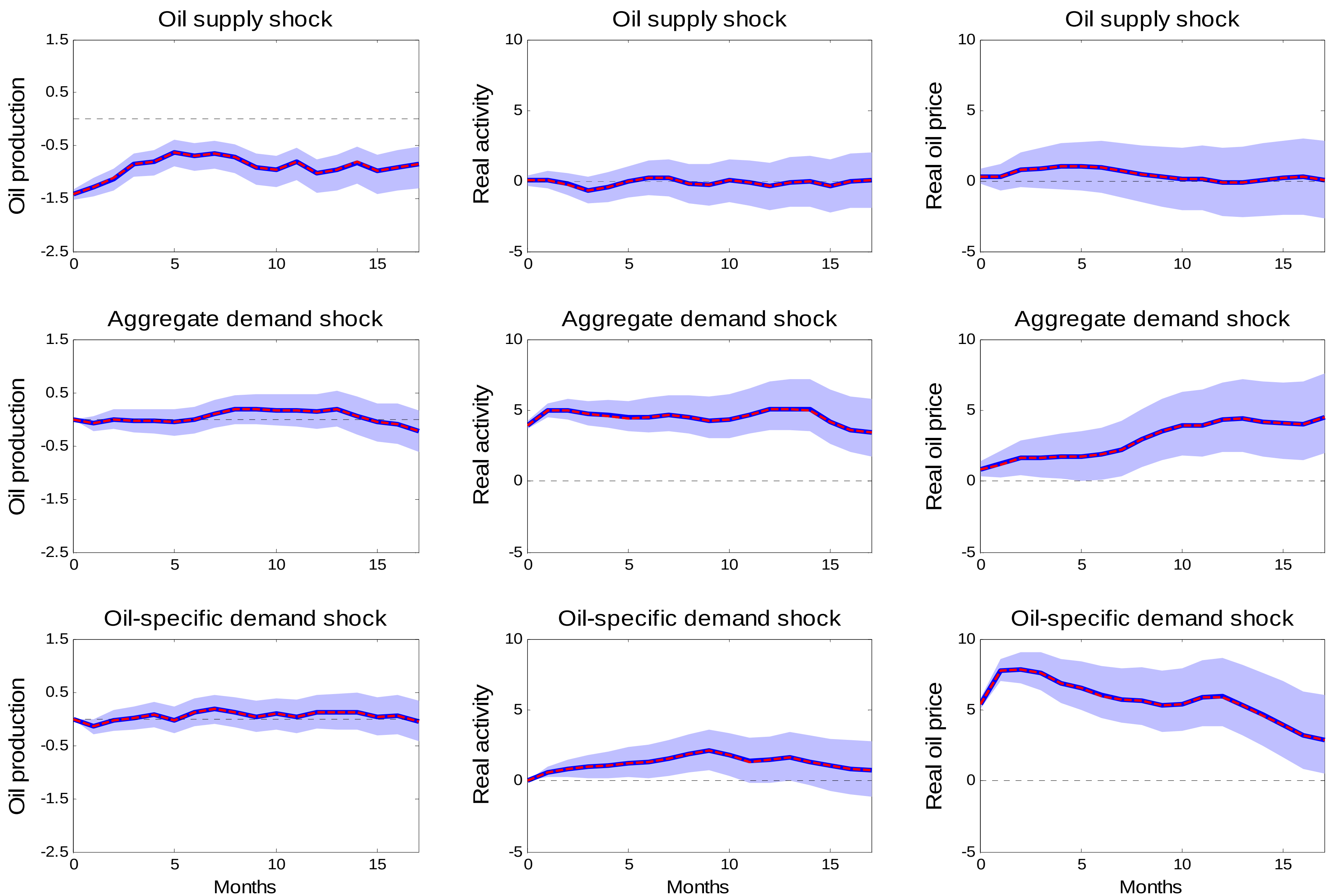
$$y_t = \alpha_{yq} q_t + \alpha_{yp} p_t + \mathbf{b}_2' \mathbf{x}_{t-1} + u_{2t}$$

inverse of oil demand curve:

$$p_t = \alpha_{pq} q_t + \alpha_{py} y_t + \mathbf{b}_3' \mathbf{x}_{t-1} + u_{3t}$$

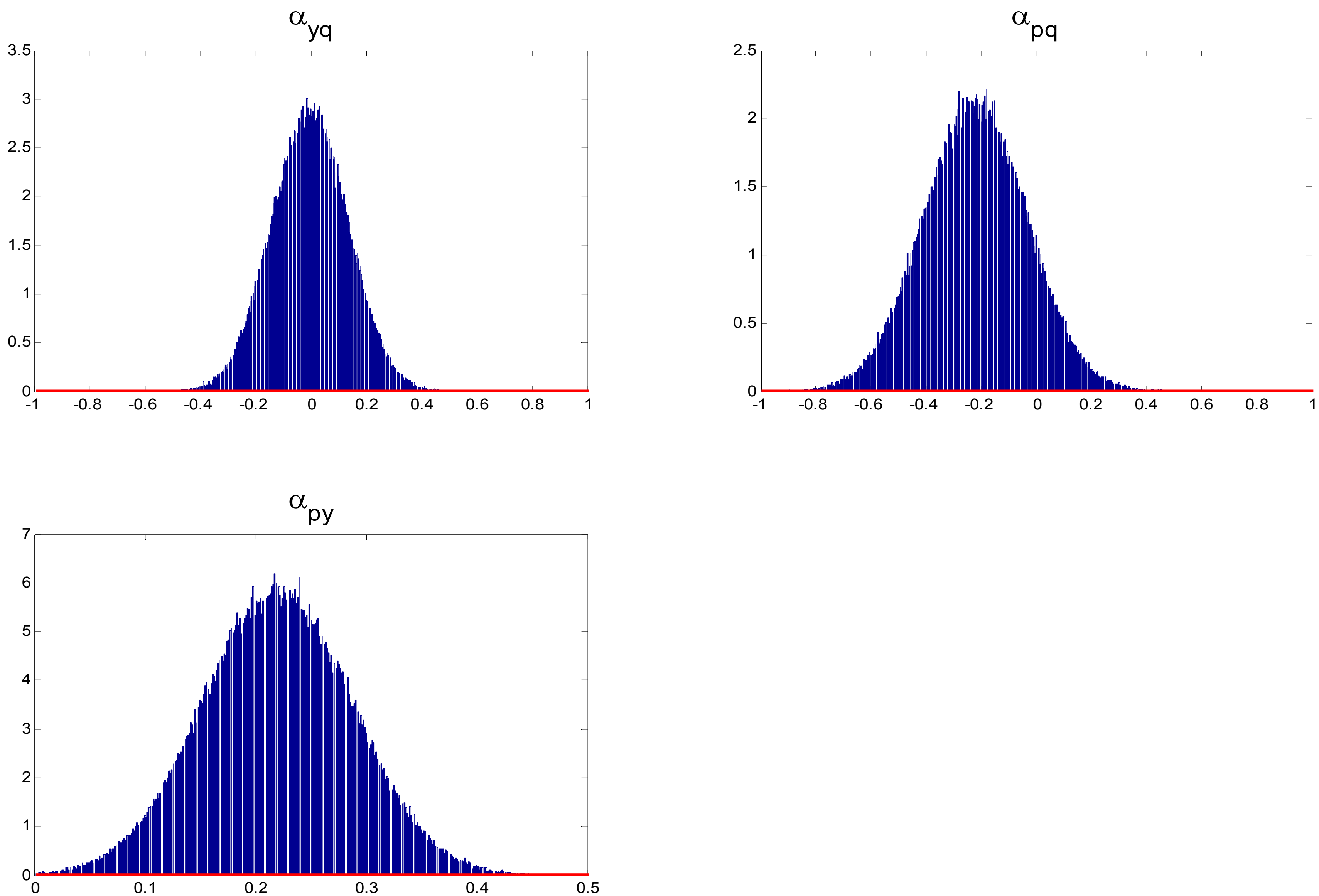
(2,1): $p(\alpha_{yq}) \sim$ Student t with location 0, scale 100, d.f. $= 3$

Same for $p(\alpha_{pq})$ and $p(\alpha_{py})$

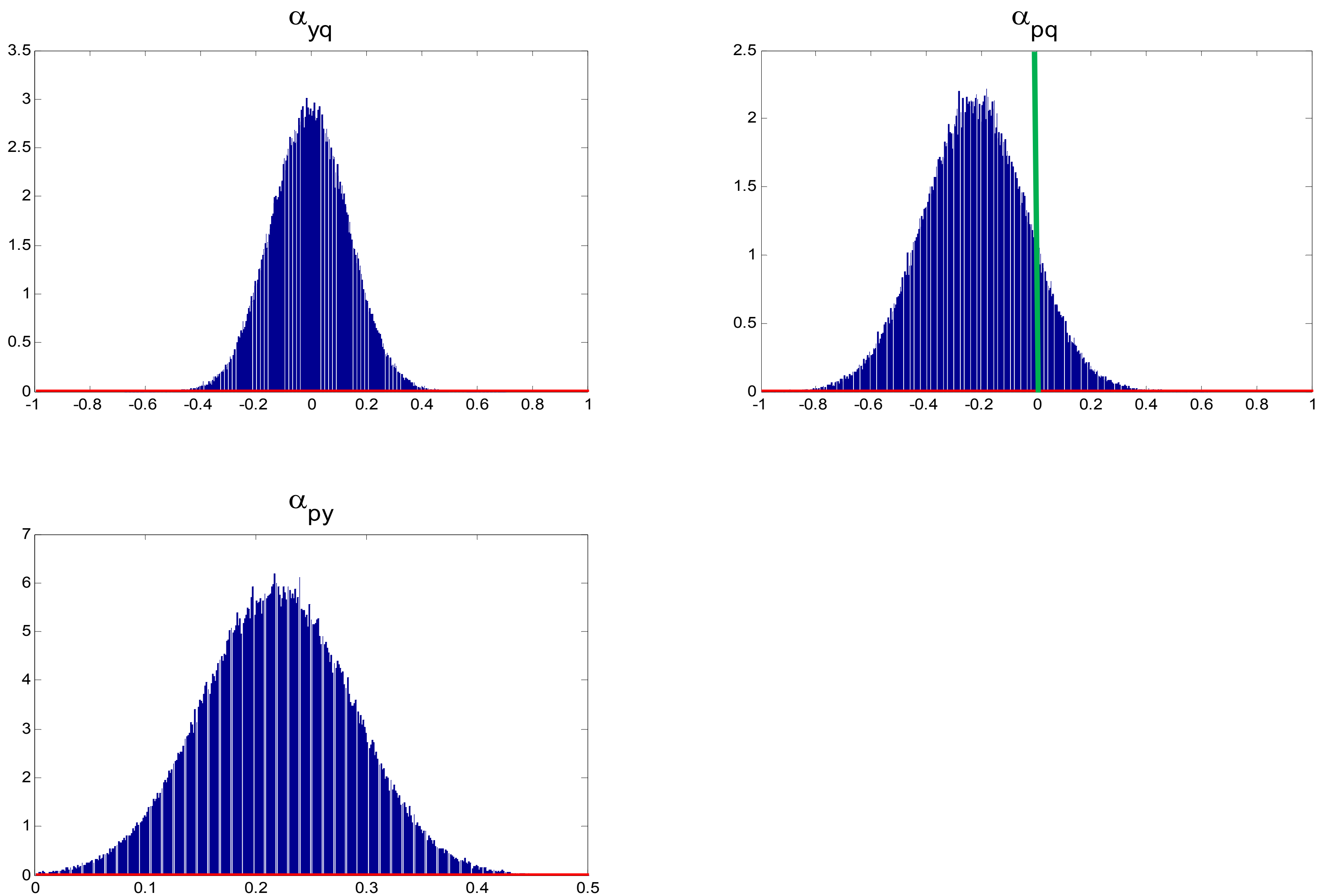


Blue: posterior median IRF as calculated using Baumeister-Hamilton algorithm for above prior.

Red: IRF calculated using Kilian’s method for original data set.

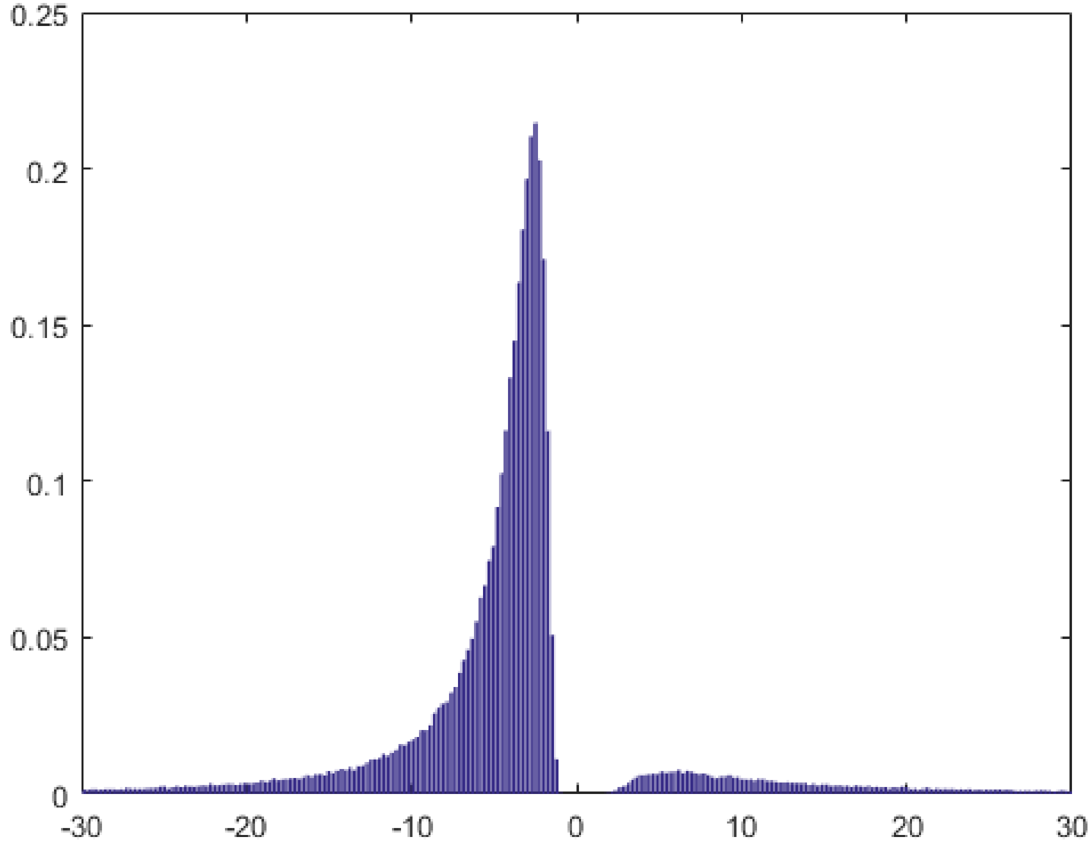


Prior (red) and posterior (blue) distributions for unknown elements of **A** (α_{pq} is reciprocal of short-run demand elasticity)



Substantial posterior probability that short-run price elasticity of demand is ±∞

Posterior density of short-run demand elasticity



12% posterior probability that demand elasticity > 0
94% posterior probability that abs(elasticity) > 2

Example 2: Kilian and Murphy (JEEA, 2012)

Partial identification using sign restrictions:

Researcher knows with certainty the signs of $\mathbf{A}^{-1}$

Further identification from hard upper bounds on certain impacts

$$\begin{bmatrix} \partial q_t/\partial u_{1t} & \partial q_t/\partial u_{2t} & \partial q_t/\partial u_{3t} \\ \partial y_t/\partial u_{1t} & \partial y_t/\partial u_{2t} & \partial y_t/\partial u_{3t} \\ \partial p_t/\partial u_{1t} & \partial p_t/\partial u_{2t} & \partial p_t/\partial u_{3t} \end{bmatrix}$$

$$= \begin{bmatrix} + & + & + \\ + & + & - \\ - & + & + \end{bmatrix}$$

oil supply ($\alpha_{qy} = 0, \alpha_{qp} > 0$)

$$q_t = \alpha_{qy} y_t + \alpha_{qp} p_t + \mathbf{b}_1' \mathbf{x}_{t-1} + u_{1t}$$

economic activity ($\alpha_{yq} = 0, \alpha_{yp} < 0$)

$$y_t = \alpha_{yq} q_t + \alpha_{yp} p_t + \mathbf{b}_2' \mathbf{x}_{t-1} + u_{2t}$$

inverse of oil demand curve ($\alpha_{pq} < 0, \alpha_{py} > 0$)

$$p_t = \alpha_{pq} q_t + \alpha_{py} y_t + \mathbf{b}_3' \mathbf{x}_{t-1} + u_{3t}$$

Under above assumptions:

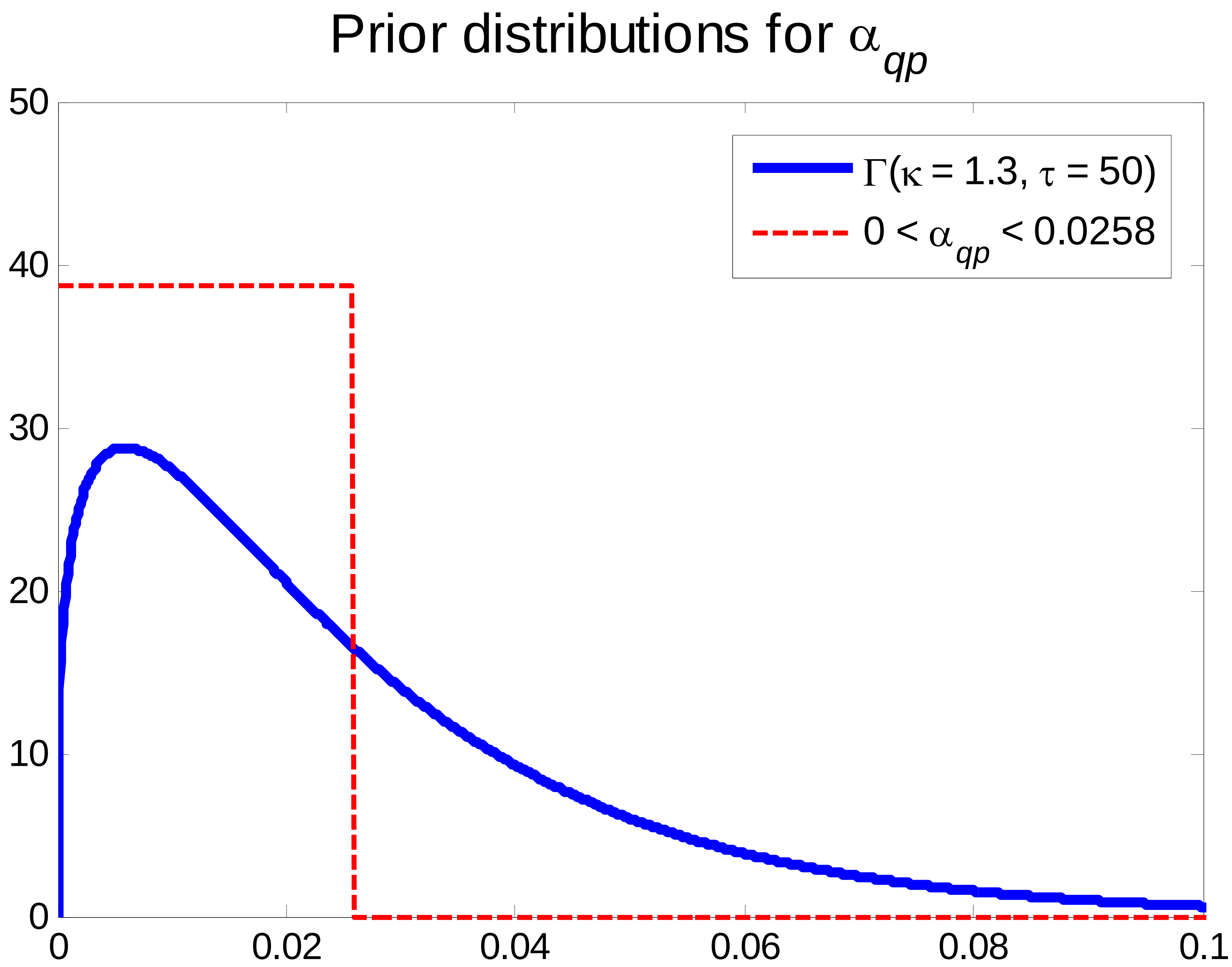
(1) Model still unidentified

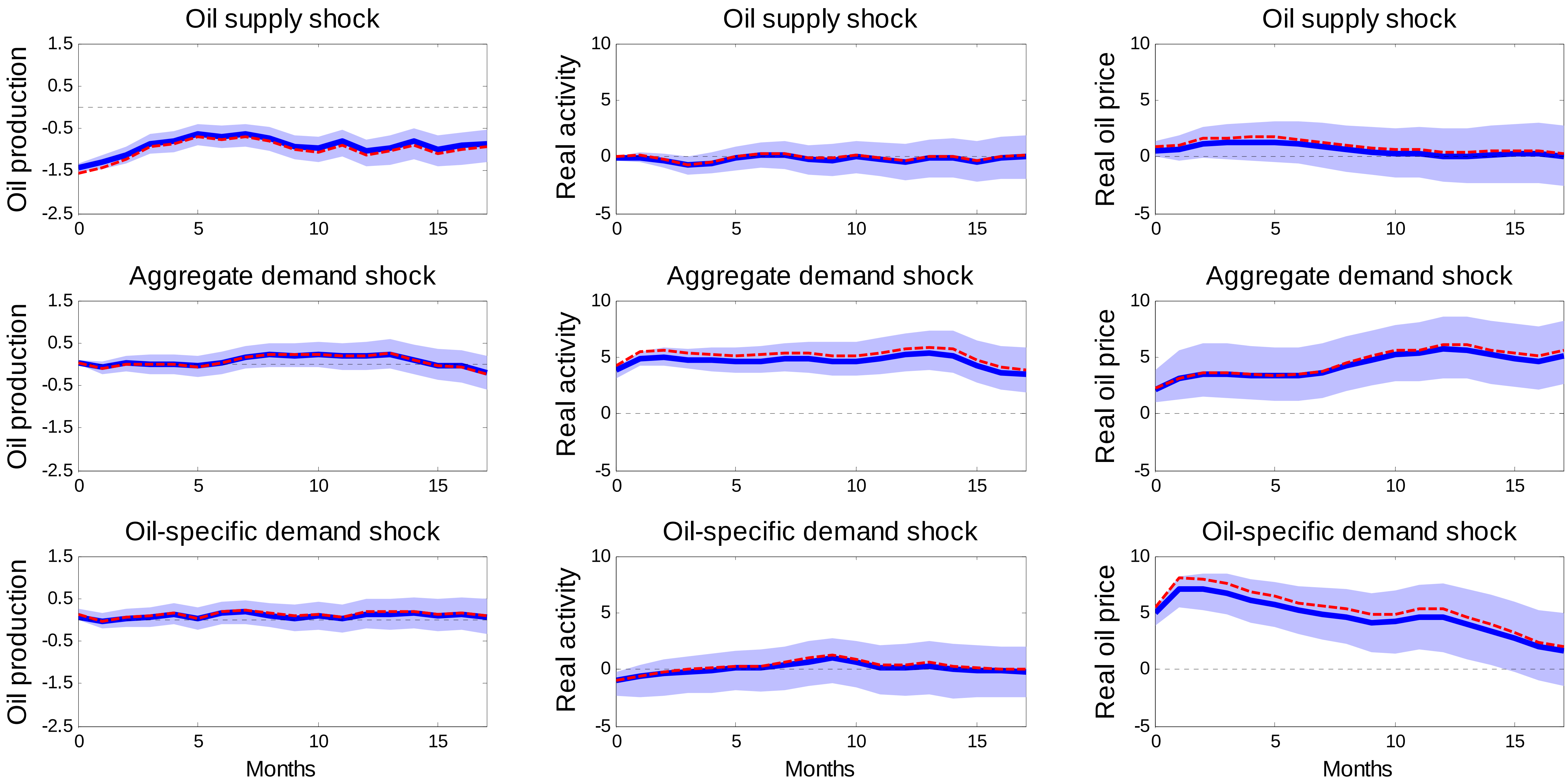
(2) $\partial \mathbf{y}_t / \partial \mathbf{u}_t'$ has desired signs for all allowable $\mathbf{A}$

Kilian and Murphy argued that
sign restrictions are not enough.

(1) Claimed we know with certainty that short-run elasticity of oil supply with respect to price is less than 0.0258:

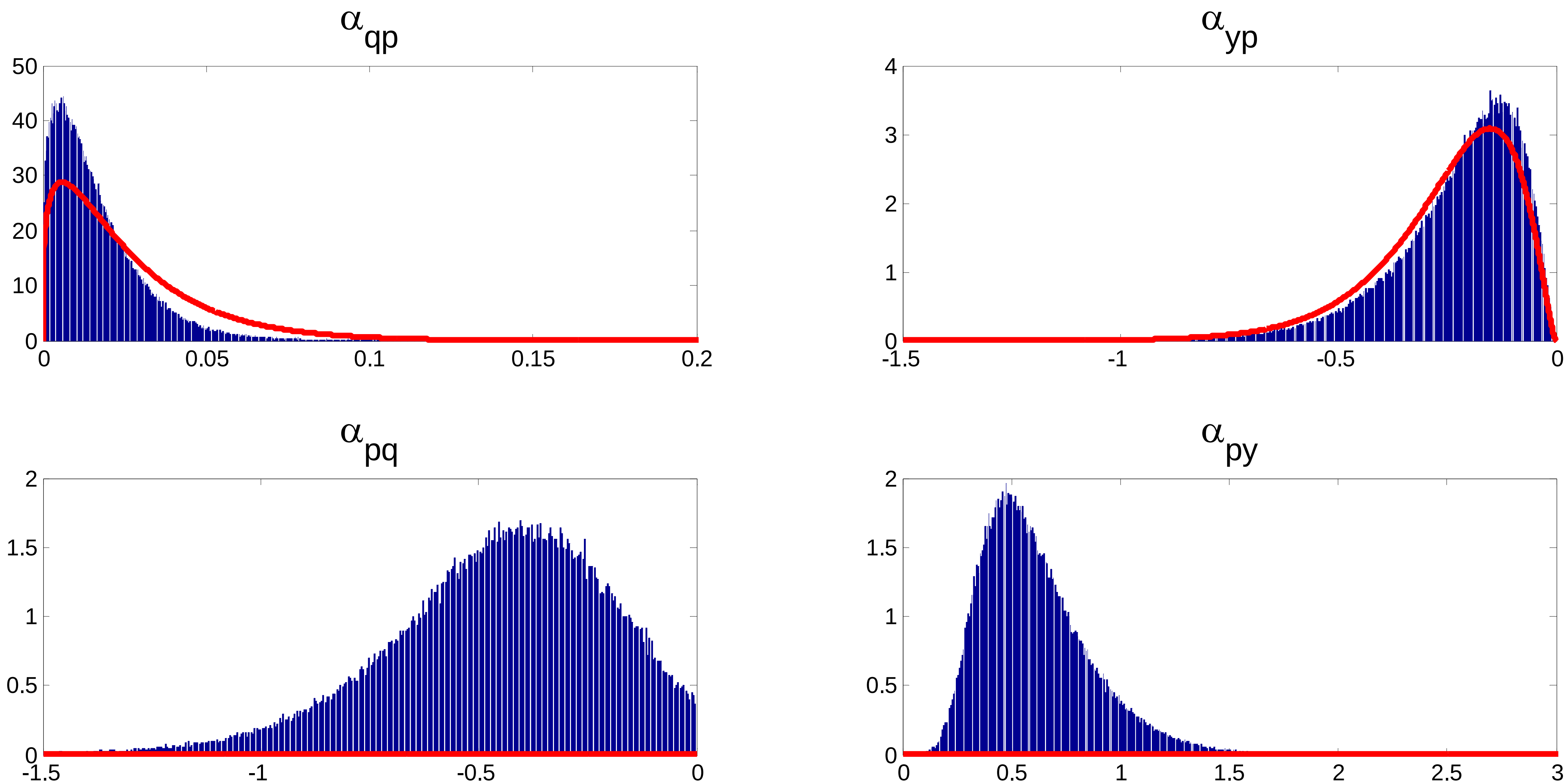
$$0 < \alpha_{qp} < 0.0258$$


Prior distributions for α_{qp}
$\Gamma(\kappa = 1.3, \tau = 50)$
$0 < \alpha_{qp} < 0.0258$
50
40
30
20
10
0
0
0.02
0.04
0.06
0.08
0.1



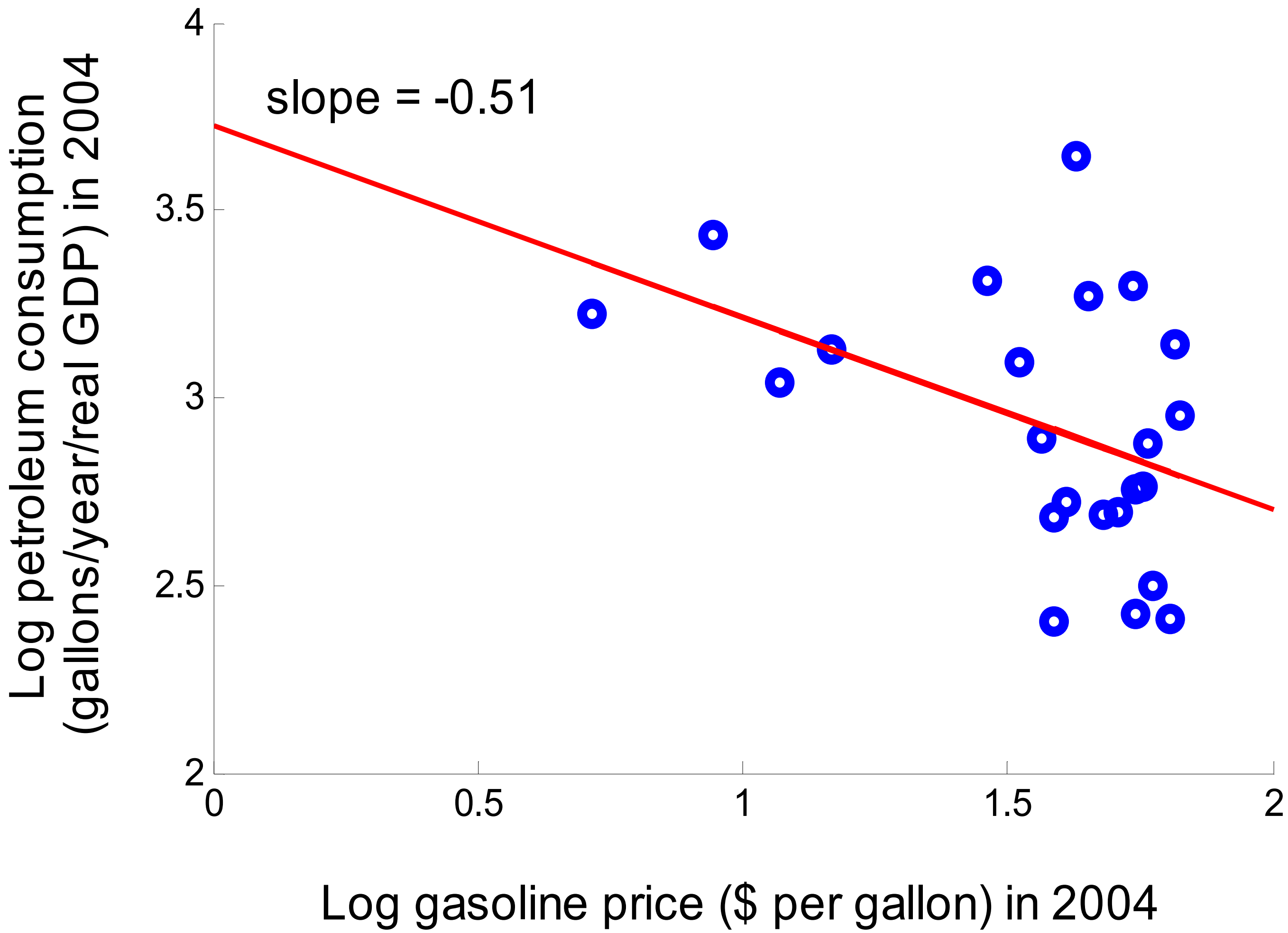
Blue: posterior median IRF as calculated using Baumeister-Hamilton algorithm for above prior.

Red: IRF calculated using Kilian-Murphy’s method for original data set.



Prior (red) and posterior (blue) distributions for unknown elements of **A**

Do we really know nothing about the price elasticity of demand?

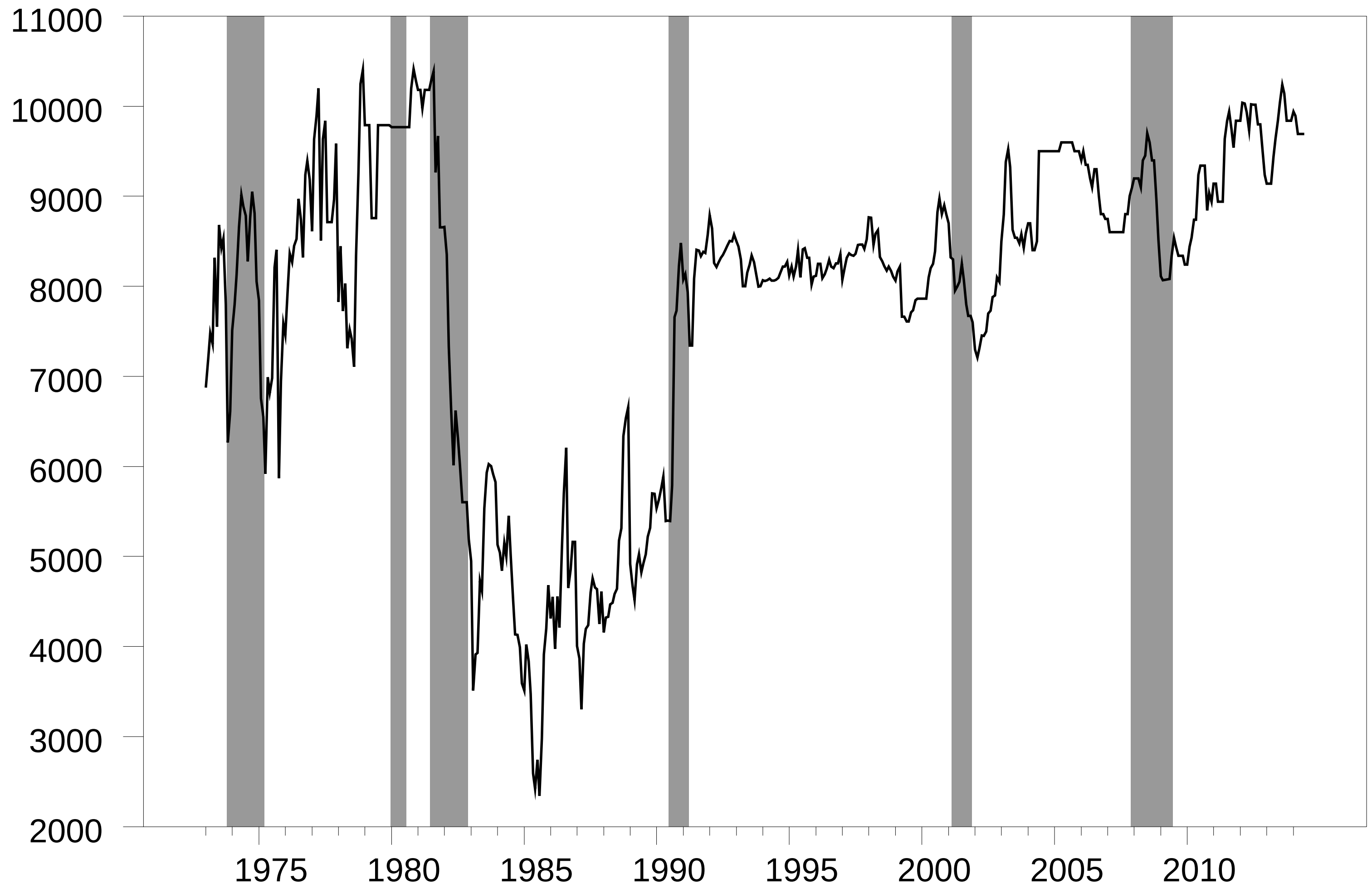


Standard error = 0.23, R^2 = 19%

- Cross-country regression of log of petroleum use per dollar of GDP on real price of gasoline for 23 OECD countries in 2004 ➡ usually interpreted as *long-run* elasticity

- Estimates from previous literature surveys:
  - Dahl and Sterner (1991): -0.86
  - Graham and Glaister (2004): -0.77
  - Brons et al. (2008): -0.84

What about the price elasticity of supply?



Saudi Arabian monthly production and U.S. recessions: significant monthly response of supply is not implausible

III. Full Bayesian treatment of the problem

A. Inventories and measurement error

$$Q_t^S - Q_t^D = \Delta I_t^*$$

$\Delta I_t^* =$ true change in inventories

$\Delta I_t =$ observed change in OECD inventories

$$q_t = 100 \ln(Q_t / Q_{t-1})$$

$$\Delta i_t^* = 100 \Delta I_t^* / Q_{t-1}$$

$$\Delta i_t = 100 \Delta I_t / Q_{t-1}$$

$$q_t = \alpha_{qp} p_t + \mathbf{b}_1' \mathbf{x}_{t-1} + u_{1t}^* \quad \text{(supply)}$$

$$y_t = \alpha_{yp} p_t + \mathbf{b}_2' \mathbf{x}_{t-1} + u_{2t}^* \quad \text{(economic activity)}$$

$$q_t = \beta_{qy} y_t + \beta_{qp} p_t + \Delta i_t^* + \mathbf{b}_3' \mathbf{x}_{t-1} + u_{3t}^* \quad \text{(demand)}$$

$$\Delta i_t^* = \psi_1^* q_t + \psi_2^* y_t + \psi_3^* p_t + \mathbf{b}_4^{*\prime} \mathbf{x}_{t-1} + u_{4t}^*$$

(inventory demand)

$$\Delta i_t = \chi \Delta i_t^* + e_t \quad \text{(inventory measurement error)}$$

Assume $\psi_2^* = 0$

Define $\psi_1 = \chi\psi_1^*,\ \psi_3 = \chi\psi_3^*,\ \rho = \frac{\chi^{-1}\sigma_e^2}{d_{33}^*+\chi^{-2}\sigma_e^2}$

System can be written in canonical form:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -\alpha_{qp} & 0 \\ 0 & 1 & -\alpha_{yp} & 0 \\ 1 & -\beta_{qy} & -\beta_{qp} & -\chi^{-1} \\ (\rho - \psi_1) & -\rho\beta_{qu} & -(\psi_3 + \rho\beta_{qp}) & (1 - \rho/\chi) \end{bmatrix}$$

III.B. Specification of prior

$$p(\mathbf{A}) = p(\alpha_{qp})p(\alpha_{yp})p(\beta_{qy})p(\beta_{qp}) \times$$
$$p(\psi_1)p(\psi_3)p(\chi)p(\rho)$$

α_{qp}: short-run supply elasticity

Student t (location 0.1, scale 0.2,

d.f. 3)

truncated ≥ 0

sensitivity analysis: $\sigma = 0.2 \rightarrow 1.0$

β_{qp}: short-run demand price elasticity

Student t (location -0.1, scale $0.2 \rightarrow 1.0$,

d.f. 3)

truncated ≤ 0

β_{qy}: short-run demand income elasticity

Student t (location 0.7, scale 0.2,

d.f. 3)

truncated ≥ 0

α_{yp}: short-run response of ind prod to p

Student t (location -0.05, scale 0.1,

d.f. 3)

truncated ≤ 0

ψ_1, ψ_3: inventory demand parameters

Student t (location 0, scale 0.5,

d.f. 3)

χ: fraction of world inventories held by OECD

Beta with mean 0.6, s.d. 0.1 ($\rightarrow 0.26$)

(OECD is 60% of world consumption)

$\rho = \frac{\chi^{-1}\sigma_e^2}{d_{33}^* + \chi^{-2}\sigma_e^2}$ (measurement error relevance)

Beta with mean 0.25, s.d. 0.12 ($\rightarrow 0.19$)

III.C. Treatment of earlier data

- Kilian (2009) and Kilian and Murphy (2012) regressions began with t = 1975:M2
- They argued that relations had changed so earlier data must all be thrown out

Could always implement Bayesian inference as follows:

(1) Find posterior distribution from a first sample with likelihood $p(\mathbf{Y}^{(1)}|\mathbf{A},\mathbf{D},\mathbf{B})$.

(2) Use this as prior for second sample.

(3) Posterior obtained in this way for second sample is identical to usual posterior from grouping all data together.

If instead replace likelihood for earlier sample with $p(\mathbf{Y}^{(1)}|\mathbf{A},\mathbf{D},\mathbf{B})^{\mu}$ for $0 \leq \mu \leq 1$ this is equivalent to using posterior means from first sample as prior means for second but with variance μ^{-1} times normal.

$\mu = 1 \Rightarrow$ earlier data given full weight

$\mu = 0 \Rightarrow$ earlier data completely discarded

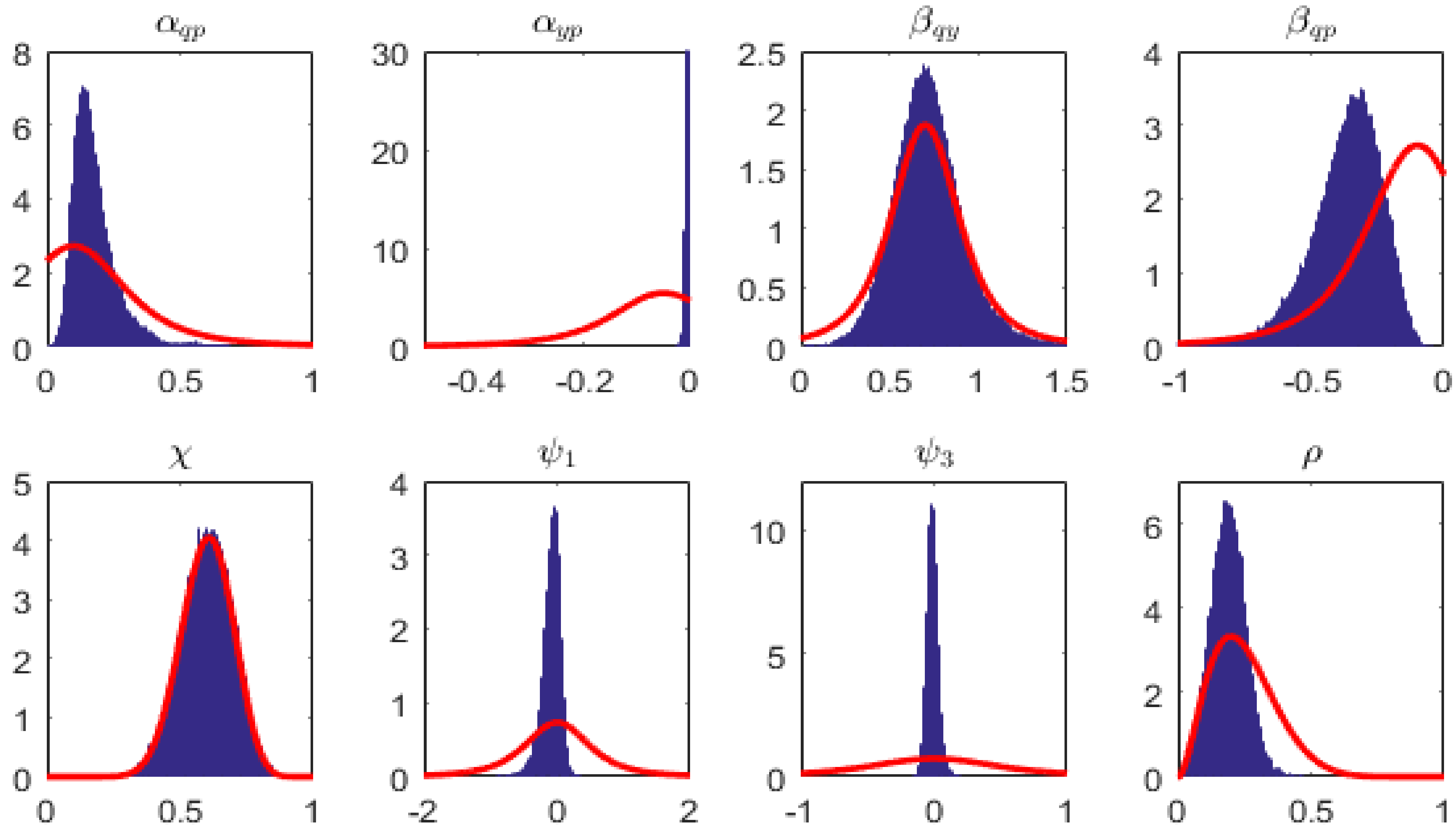
$\mu = 0.5$ used for baseline results

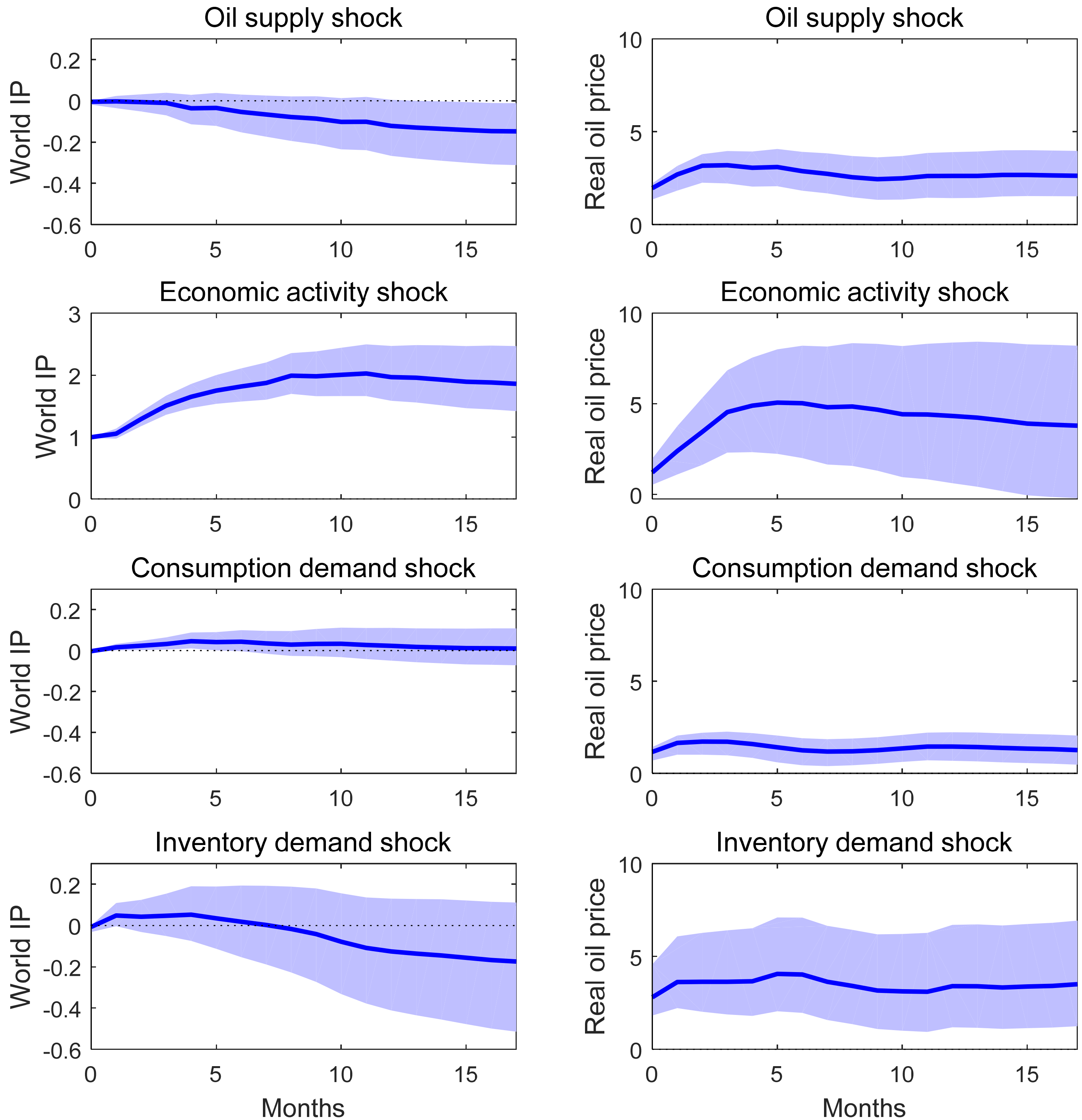
$\mu = 0.25$ for sensitivity analysis

IV. Empirical results

A. Baseline results

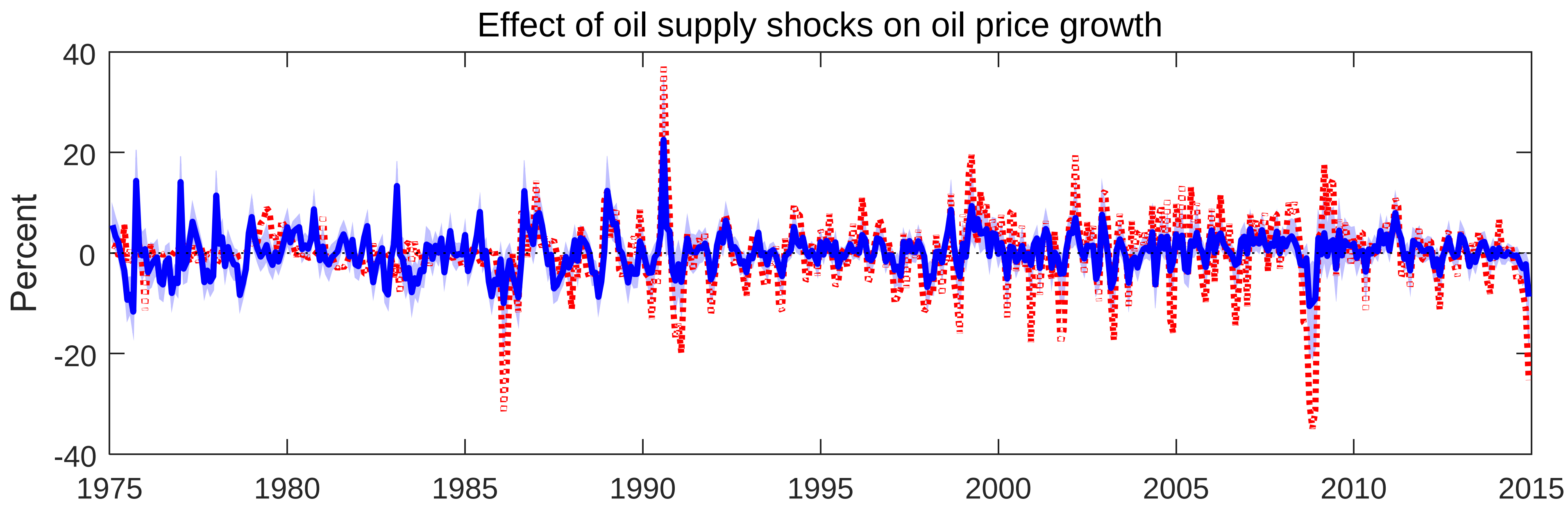
Prior (red) and posterior (blue) distributions for unknown elements of **A**



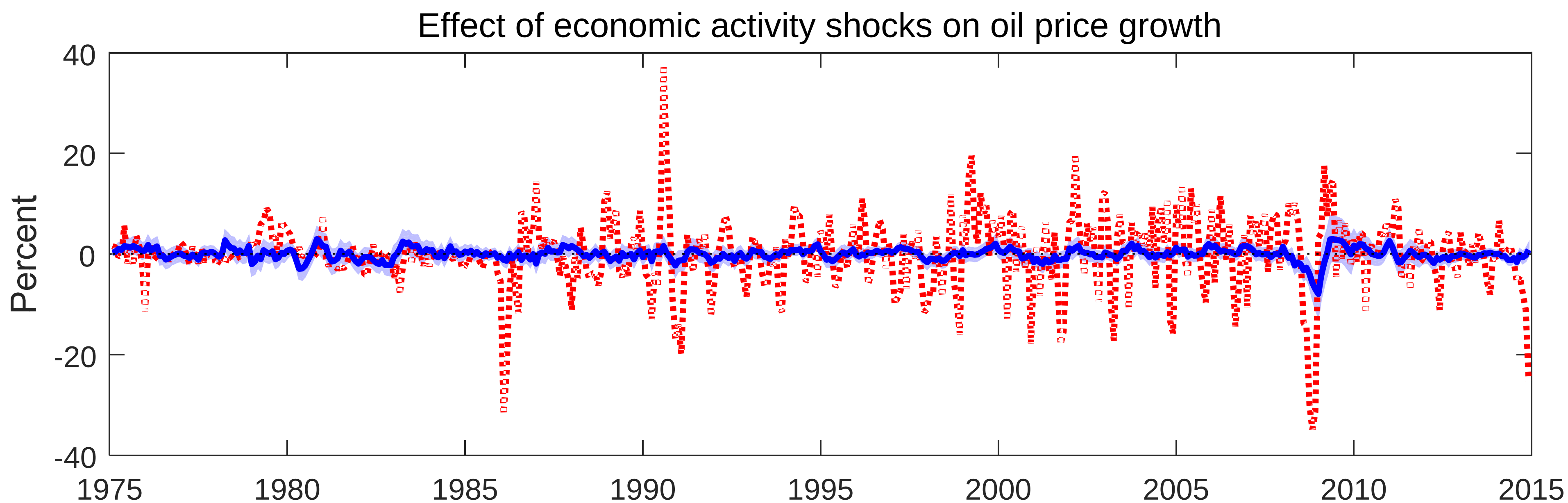

Oil supply shock
World IP
0.2
0
-0.2
-0.4
-0.6
0
5
10
15
Oil supply shock
Real oil price
10
5
0
Economic activity shock
World IP
3
2
1
0
Economic activity shock
Real oil price
Consumption demand shock
World IP
Consumption demand shock
Real oil price
Inventory demand shock
World IP
Inventory demand shock
Real oil price
Months
Months

Posterior medians and 95% credibility sets for various magnitudes of interest

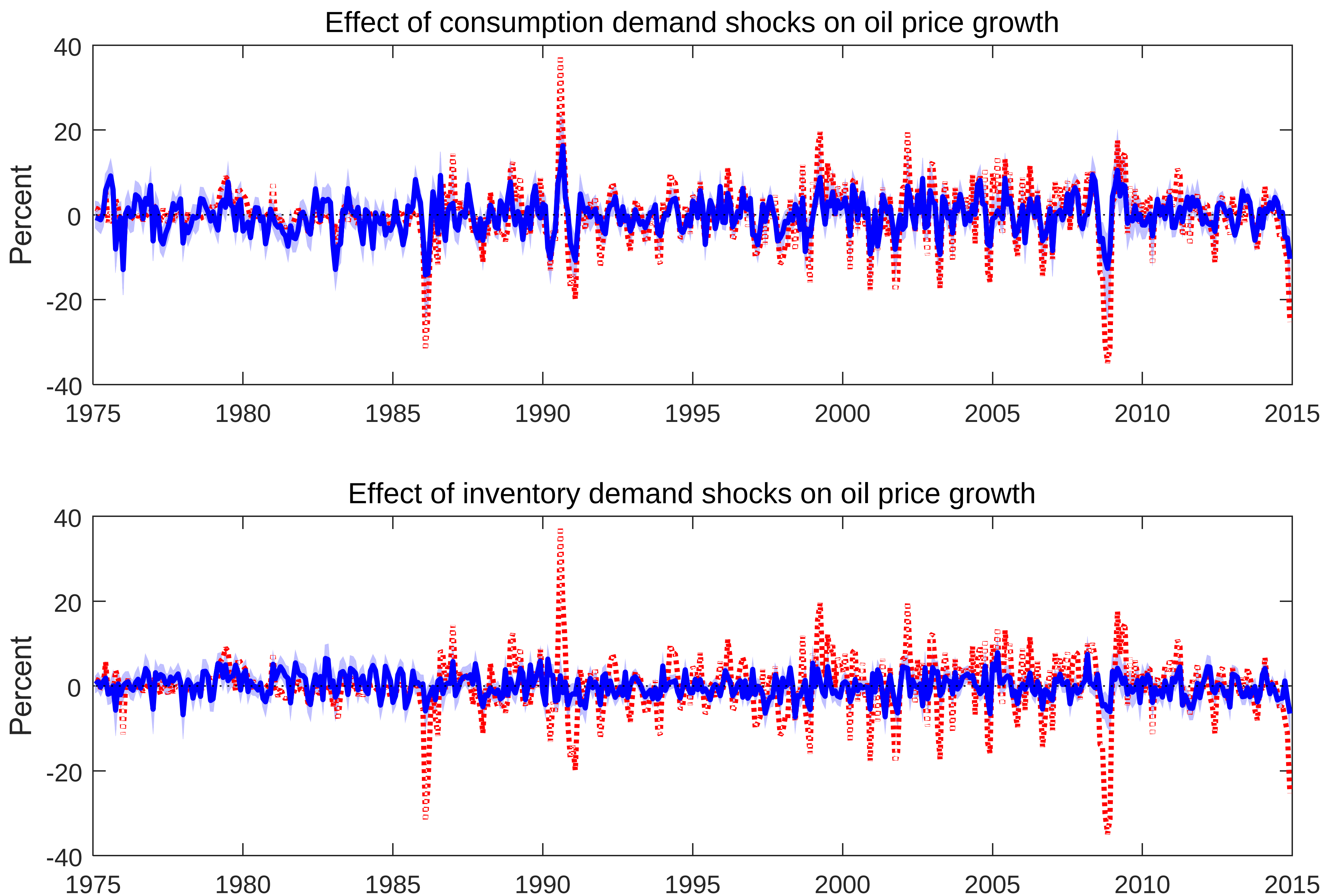
Short-run price elasticity of oil supply α_{qp}	**0.16**
	(0.07, 0.38)
Short-run price elasticity of oil demand β_{qp}	**-0.35**
	(-0.71, -0.16)
Effect of oil supply shock that raises real oil price 1% on y_{t+12}	**-0.06**
	(-0.15, 0.00)
Effect of oil demand shock that raises real oil price 1% on y_{t+12}	**0.02**
	(-0.04, 0.12)
Effect of inventory shock that raises real oil price 1% on y_{t+12}	**-0.04**
	(-0.14, 0.05)



Effect of oil supply shocks on oil price growth
Percent
40
20
0
-20
-40
1975
1980
1985
1990
1995
2000
2005
2010
2015



Effect of economic activity shocks on oil price growth
Percent
40
20
0
-20
-40
1975
1980
1985
1990
1995
2000
2005
2010
2015



Effect of consumption demand shocks on oil price growth
Percent
40
20
0
-20
-40
1975
1980
1985
1990
1995
2000
2005
2010
2015
Effect of inventory demand shocks on oil price growth
Percent
40
20
0
-20
-40
1975
1980
1985
1990
1995
2000
2005
2010
2015

Posterior medians and 95% credibility sets for contribution of shocks to various historical episodes

Historical episode	Actual real oil price growth (RAC)	Benchmark
(1)	(2)	(3)
Jan-July 1986 (*supply*)	-87.65	**-40.29** [46%] (-70.54, -15.31)
Jan-June 2008 (*consumption*)	38.86	**21.00** [54%] (7.53, 37.72)
July-Dec 2014 (*supply*)	-55.55	**-17.54** [32%] (-35.17, -5.15)

Decomposition of variance of 12-month-ahead forecast errors

	Oil production	Industrial production	Real oil price	Inventories
Oil supply	**1.61** [64%] (0.76, 2.17)	**0.01** [3%] (0.00, 0.03)	**17.23** [36%] (6.46, 34.55)	**0.07** [4%] (0.03, 0.12)
Economic activity	**0.09** [3%] (0.04, 0.15)	**0.34** [90%] (0.30, 0.40)	**2.16** [4%] (0.88, 4.44)	**0.05** [3%] (0.02, 0.09)
Oil consumption demand	**0.46** [19%] (0.18, 0.84)	**0.02** [4%] (0.01, 0.03)	**20.16** [42%] (6.91, 35.50)	**0.57** [38%] (0.25, 0.96)
Speculative oil demand	**0.35** [14%] (0.12, 0.96)	**0.01** [3%] (0.00, 0.02)	**8.86** [18%] (4.98, 14.24)	**0.82** [54%] (0.47, 1.16)

Sensitivity analysis: historical decomposition of selected episodes

Historical episode	Actual real oil price growth (RAC)	Benchmark	Supply and demand elasticities	Measurement error	Pre-1975 data	Lagged structural coefficients	Variances of shocks
		(1)	(2)	(3)	(4)	(5)	(6)
Jan-July 1986 (*supply*)	-87.65	**-40.29** [46%] (-70.54, -15.31)	**-27.90** [32%] (-73.12, -5.29)	**-40.38** [46%] (-73.04, -16.89)	**-40.11** [46%] (-70.24, -16.28)	**-41.85** [48%] (-71.54, -17.22)	**-40.62** [46%] (-72.48, -16.33)
Jan-June 2008 (*consumption*)	38.86	**21.00** [54%] (7.53, 37.72)	**27.82** [72%] (7.22, 45.60)	**21.20** [55%] (2.95, 38.57)	**22.47** [58%] (8.35, 39.22)	**20.42** [53%] (7.47, 36.45)	**20.86** [54%] (6.97, 37.15)
July-Dec 2014 (*supply*)	-55.55	**-17.54** [32%] (-35.17, -5.15)	**-11.23** [20%] (-36.53, -1.10)	**-17.57** [32%] (-36.75, -5.80)	**-17.71** [32%] (-35.08, -5.80)	**-17.14** [31%] (-34.39, -5.29)	**-17.72** [32%] (-36.43, -5.61)

Sensitivity analysis: variance decomposition of real oil price

Benchmark	Supply and demand elasticities	Measure-ment error	Pre-1975 data	Lagged structural coefficients	Variances of shocks	Replace RAC with WTI
(1)	(2)	(3)	(4)	(5)	(6)	(7)
(6) *12-month-ahead MSE of oil price due to oil supply shock*						
17.23 [36%]	**13.05** [27%]	**15.33** [32%]	**17.10** [35%]	**17.42** [36%]	**17.18** [35%]	**19.15** [35%]
(6.46, 34.55)	(4.26, 33.36)	(5.20, 32.89)	(6.42, 34.83)	(7.54, 33.48)	(6.51, 34.07)	(7.66, 42.43)
(7) *12-month-ahead MSE of oil price due to economic activity shock*						
2.16 [4%]	**2.18** [5%]	**2.13** [4%]	**2.32** [5%]	**2.42** [5%]	**2.16** [4%]	**2.72** [5%]
(0.88, 4.44)	(0.90, 4.50)	(0.86, 4.42)	(0.93, 4.86)	(1.05, 4.83)	(0.89, 4.45)	(1.22, 5.14)
(8) *12-month-ahead MSE of oil price due to oil consumption demand shock*						
20.16 [42%]	**24.85** [51%]	**22.91** [48%]	**20.60** [42%]	**20.05** [41%]	**20.21** [42%]	**22.78** [41%]
(6.91, 35.50)	(7.41, 39.95)	(6.27, 38.63)	(7.18, 35.90)	(7.70, 33.98)	(7.15, 35.34)	(5.71, 39.30)

Conclusions

(1) Structural interpretation of correlations only possible by drawing on prior understanding of economic structure

(2) Doing so using the formal Bayesian statistical theory offers many advantages over existing approaches

(3) Application to oil supply-demand example confirms conclusions from other methods
   (a) an oil price increase that results from decrease in supply reduces economic activity
   (b) an oil price increase that results from increase in demand does not
   (c) demand shocks were important historically and particularly in 2008:H1 and 2008:H2

End of main slides

More details on Kilian-Murphy (2012)

Structural model:

$$\mathbf{A}\mathbf{y}_t = \boldsymbol{\lambda} + \mathbf{B}_1\mathbf{y}_{t-1} + \cdots + \mathbf{B}_m\mathbf{y}_{t-m} + \mathbf{u}_t$$

Reduced form:

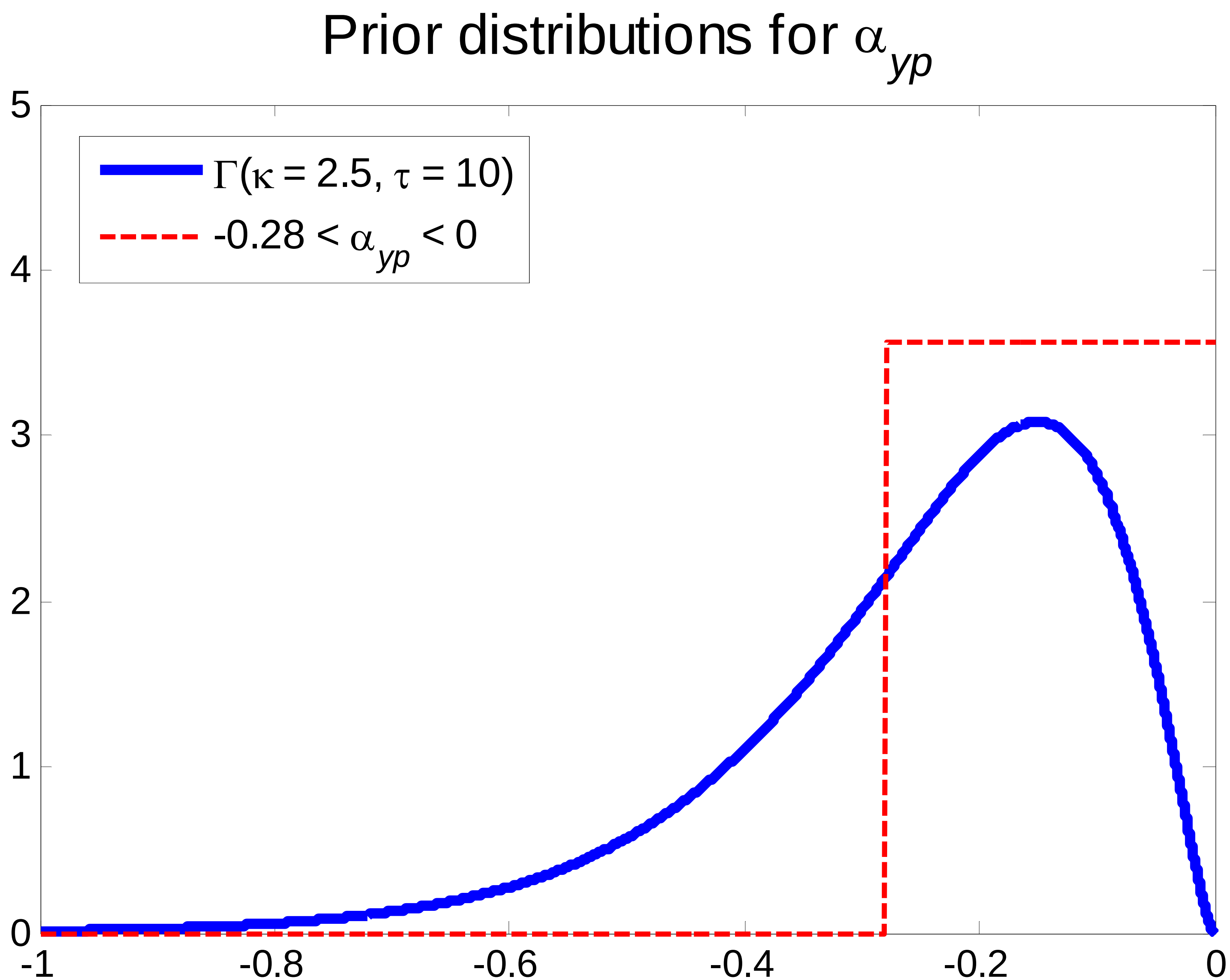
$$\mathbf{y}_t = \mathbf{c} + \boldsymbol{\Phi}_1\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Phi}_p\mathbf{y}_{t-m} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t = \mathbf{A}^{-1}\mathbf{u}_t$$

$$\frac{\partial \mathbf{y}_t}{\partial \mathbf{u}_t'} = \mathbf{A}^{-1}$$

Kilian and Murphy argued that sign restrictions are not enough.

(2) Claimed we know with certainty that the product of (2,3) element of $\mathbf{A}^{-1}$ with d_{33} falls in $[-1.5, 0]$



Prior distributions for α_{yp}
$\Gamma(\kappa = 2.5, \tau = 10)$
$-0.28 < \alpha_{yp} < 0$
5
4
3
2
1
0
-1
-0.8
-0.6
-0.4
-0.2
0

Prior for **A**:

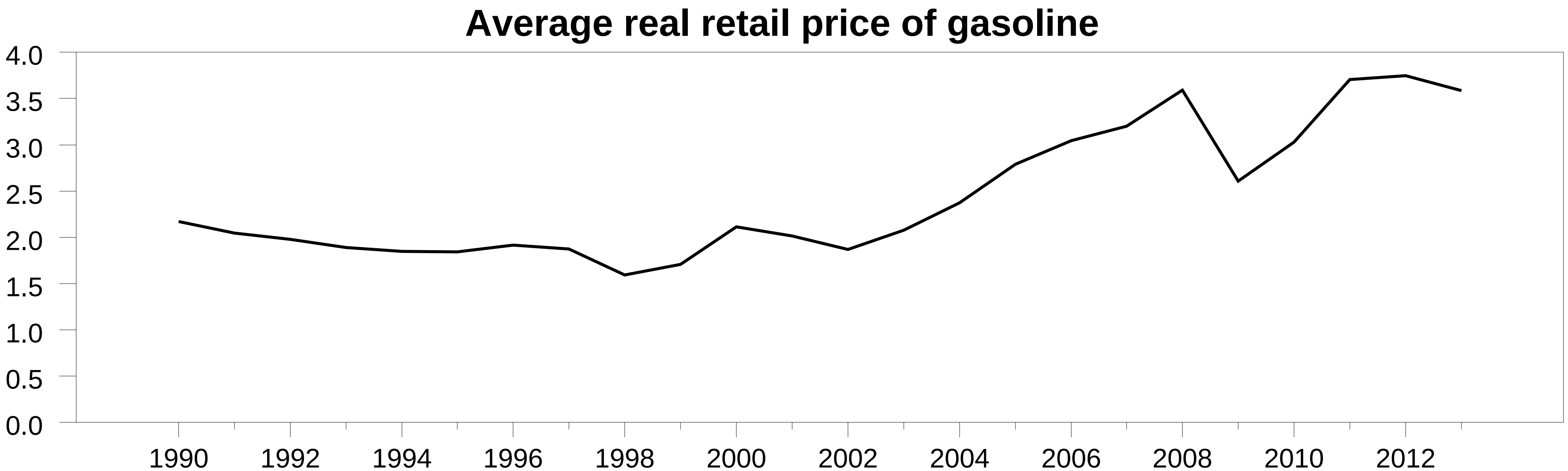
$\alpha_{qy} = \alpha_{yq} = 0$

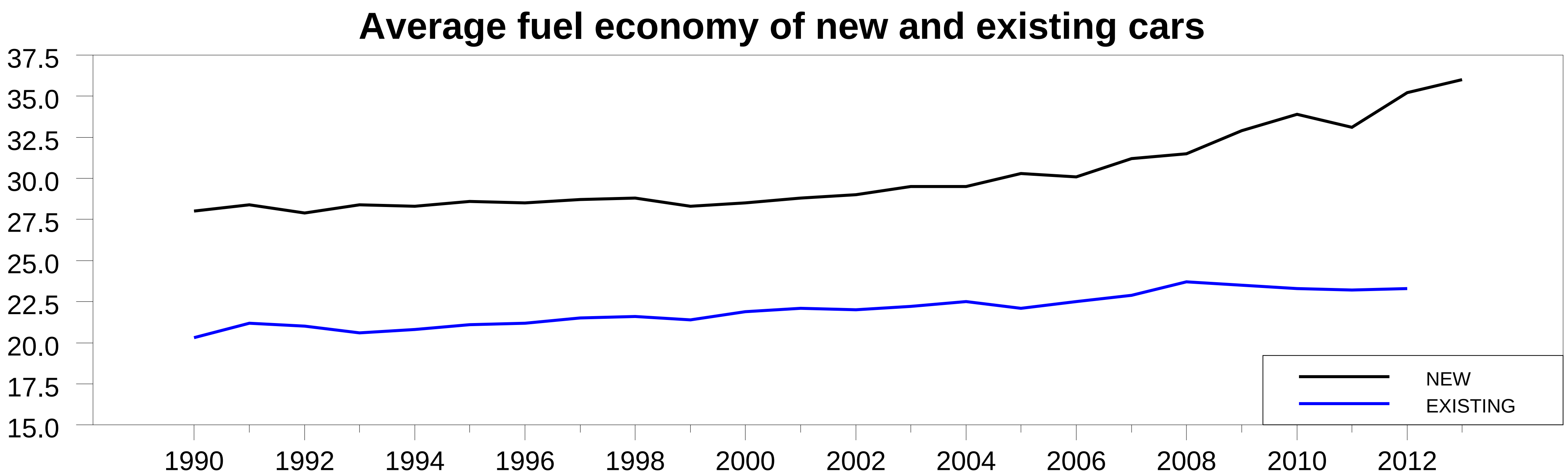
$\alpha_{qp} \sim \Gamma(1.3, 50)$

$-\alpha_{yp} \sim \Gamma(2.5, 10)$

$\alpha_{pq} \sim$ Student t (0,100,3) truncated to be negative

$\alpha_{py} \sim$ Student t (0,100,3) truncated to be positive





We also know short-run elasticity is much less than long-run

Recommended default priors (Minnesota prior)

Doan, Litterman, Sims (1984)

Sims and Zha (1998)

- elements of $\mathbf{m}_i$ corresponding to

lag 1 given by $\mathbf{a}_i$

- all other elements of $\mathbf{m}_i$ are zero
- $\mathbf{M}_i$ diagonal with smaller values on

bigger lags

$\Rightarrow$ prior belief that each element of

$\mathbf{y}_t$ behaves like a random walk

τ_i function of $\mathbf{A}$ (or prior mode of $p(\mathbf{A})$) and scale of data prior: $p(\mathbf{A})$

posterior: $p(\mathbf{A}|\mathbf{Y}_T) = \dfrac{k_T p(\mathbf{A})[\det(\mathbf{A}\hat{\boldsymbol{\Omega}}_T\mathbf{A}')]^{T/2}}{\prod_{i=1}^{n}[(2\tau_i/T)+(\zeta_i^*/2T)]^{\kappa_i+T/2}}$

$k_T =$ constant that makes this integrate to 1

If $\mathbf{M}_i^{-1} = \mathbf{0}$, and $\tau_i = \kappa_i = 0$,

$$p(\mathbf{A}|\mathbf{Y}_T) = \frac{k_T p(\mathbf{A})|\det(\mathbf{A}\hat{\boldsymbol{\Omega}}_T\mathbf{A}')|^{T/2}}{\left\{\det\left[\text{diag}(\mathbf{A}\hat{\boldsymbol{\Omega}}_T\mathbf{A}')\right]\right\}^{T/2}}$$

$$p(\mathbf{A}|\mathbf{Y}_T) = \frac{k_T p(\mathbf{A}) |\det(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')|^{T/2}}{\left\{\det\left[\text{diag}(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')\right]\right\}^{T/2}}$$

If evaluate at any $\mathbf{A}$ for which

$\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}' = \text{diag}(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')$, then

$$p(\mathbf{A}|\mathbf{Y}_T) = k_T p(\mathbf{A})$$

$$p(\mathbf{A}|\mathbf{Y}_T) = \frac{k_T p(\mathbf{A})|\det(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')|^{T/2}}{\left\{\det\left[\text{diag}(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')\right]\right\}^{T/2}}$$

If evaluate at any $\mathbf{A}$ for which $\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}' \neq \text{diag}(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')$, then

$$\det\left[\text{diag}(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')\right] \geq \det(\mathbf{A}\hat{\mathbf{\Omega}}_T\mathbf{A}')$$

by Hadamard's inequality and

$p(\mathbf{A}|\mathbf{Y}_T) \to 0$ as $T \to \infty$

Model is just-identified if there is only one matrix $\mathbf{A}$ for which $p(\mathbf{A}) > 0$ and $\mathbf{A}\boldsymbol{\Omega}_0\mathbf{A}' = \text{diag}(\mathbf{A}\boldsymbol{\Omega}_0\mathbf{A}')$ for $\boldsymbol{\Omega}_0 = E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')$

If model is just-identified, then $p(\mathbf{A}|\mathbf{Y}_T)$ collapses to a point mass at $\mathbf{A}_0$

Model is under-identified if there is more than one matrix $\mathbf{A}$ for which $p(\mathbf{A}) > 0$ and $\mathbf{A}\mathbf{\Omega}_0\mathbf{A}' = \text{diag}(\mathbf{A}\mathbf{\Omega}_0\mathbf{A}')$

If model is under-identified, then $p(\mathbf{A}|\mathbf{Y}_T)$ is asymptotically confined to set $S(\mathbf{\Omega}_0) = \{\mathbf{A} : \mathbf{A}\mathbf{\Omega}_0\mathbf{A}' = \text{diag}(\mathbf{A}\mathbf{\Omega}_0\mathbf{A}')\}$

Proposal: if not certain about some
of the identifying assumptions, represent
that uncertainty in form of $p(\mathbf{A})$.
Some of this uncertainty will remain (and
be included in any posterior statements)
even if sample size T goes to ∞.

Effects of relaxing sign constraint on α_{yp}