The Stock Market in an Inflation-Targeting Economy*

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Abstract

We construct recursive solutions for, and study quasi-explicitly the properties of the dynamic equilibrium of an economy with three types of agents: (i) household/investors who supply labor with a finite elasticity, consume a large variety of goods that are not perfect substitutes and trade government bonds; (ii) firms that produce those varieties of goods, setting prices in a Calvo manner; (iii) a government that collects an exogenous fiscal surplus and acts mechanically, buying and selling bonds in accordance with a Taylor policy rule based on expected inflation. In this equilibrium, we price the stock market, defined as the present discounted value of firms' profits and simulate the joint behavior of stock returns and inflation. We use the simulated data to gauge the adequacy of the model in comparison with empirical stylized facts.

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> Fama and Schwert (1977) found that expected stock returns did not increase one-for-one with inflation. They interpreted this result to say that expected returns are higher in bad economic times, since people are less willing to hold risky assets, and are lower in good times. Inflation is lower in bad times and higher in good times, so lower expected returns in times of high inflation are not a result of inflation, but a coincidence.
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> Cochrane (2005b)

Judging from the attention paid by stock market traders to utterances of the governors of central banks, one may suppose that there exists a strong transmission mechanism linking the stock market to monetary policy. Clearly, investors in that market make every effort to anticipate every move of the central bank. Yet, the literature in monetary economics by and large ignores the stock market and, given the complementary focus on fiscal policy, is only interested in pricing nominal government bonds, a task which is accomplished by means of the private sector's Euler condition of portfolio choice (known in this context as the Fisher equation). There have been several exceptions (Marshall (1992), Challe and Giannitsarou (2014), Swanson (2014)) and a few attempts to relate monetary economics to financial economics. Several papers going in that direction are: Svensson (1989), Benhabib, Schmitt-Grohe and Uribe (2001, 2002), Nakajima and Polemarchakis (2005) and Magill and Quinzii (2009, 2012). We draw inspiration from these papers, our purpose being to describe, as quantitatively as possible, the key features of the relation between monetary and fiscal policy on the one hand and the stock market on the other.

The main and most useful result of the model will be the manner in which the stochastic process of equilibrium securities prices corresponds to a given monetary-policy process.[1] It will indicate how any multi-period government behavior is transmitted dynamically to financial markets. Investors are interested in that transmission because, among other things, they would very much like to know *whether shares of stock are a good hedge against inflation*. Conversely, the knowledge of that transmission can guide central bankers in their attempt to utilize information from financial markets to gauge anticipations of monetary policy (Bernanke and Gertler (1999, 2001), Bernanke and Kuttner (2004)).

A financial economist, building on monetary and fiscal policy research, cannot treat the government as just any other trader that seeks to optimize his/her lifetime utility function under some budget constraint. Indeed, most of the work in that area attributes no explicit objective function to the government.[2] Issues of feasibility, stability and determinacy are discussed at length, but the objective

[1] See Asness (2003).

[2] Three strands of monetary economics provide exceptions. First, ad hoc mean-variance objective functions are used to justify the linear Taylor rule (Woodford (2003), pages 535ff). Second, in the context of incomplete markets where nominal assets are traded, some researchers (e.g., Chari et al. (1993), Allen et al. (2012)) ask: can monetary policy maximize welfare by serving to render the market more complete. The optimal policy involves unrealistically volatile inflation rate and nominal interest rate. Third, Ramsey-optimal inflationary taxes have been derived by e.g., Persson et al. (1987).

functions of the government and the central bank are not stated explicitly. Instead, a behavior rule is postulated as a quasi-mechanical intervention formula. Today, the majority of central banks follows a policy called "inflation targeting" epitomized by the famous Taylor (1993) or Henderson-McKibbin (1993) rule.

Following Sargent and Wallace (1975), the literature has stressed the unavoidable financial linkage between monetary and fiscal policies. Indeed, another distinction between the government and regular investors arises in the specification of its income. In an exchange economy, regular investors receive an income, which is exogenous. Or, in a production economy, they may draw some income from their labor and that income is dictated by the production function, which is specified *ab initio*. The government is different in that it draws income from taxes. A major distinction must be drawn between a specification in which the budget surplus of the government is exogenous – a so-called "non Ricardian" fiscal policy – and one in which it will at some point or the other have to raise enough taxes to repay its debt – a "Ricardian" policy. The distinction between Ricardian and non-Ricardian fiscal regimes can be traced back to Aiyagari and Gertler (1985), Leeper (1991) and Canzoneri et al. (2011). In this paper, we assume that fiscal policy is non Ricardian. Under a non Ricardian policy, it is conceivable for the government in some sense to default but we do not model that event. More importantly for our purposes, in a non Ricardian regime, some of the debt may be monetized.

When setting the nominal rate of interest, the principal aim of the central bank is to anchor inflationary expectations. In most models of monetary economics, the Taylor rule is backward looking in that it captures the central bank's reaction to *realized inflation*. Realized inflation is really a proxy for rationally expected inflation, a proxy that a central bank would rely on when it has access to incomplete information. In this paper, we make the assumption that the central bank has access to the same full information as the private sector. For that reason, we write the Taylor rule as a forward-looking formula relating the nominal rate of interest to the *rationally-expected rate of inflation*, as in Clarida et al. (2000), Bernanke and Bovin (2000).[3]

One more specification differentiates the present paper from most work in monetary economics. We postulate a finite terminal date for the economy although, stepping backward, we are able to postpone it indefinitely until we reach an unchanging solution. The private agents' utility functions do not extend beyond the terminal date and, after that date, the prices of all securities, including the agents' financial wealth, are set equal to zero, both in real and in nominal terms.[4] In this way, the enforcement of terminal conditions is facilitated and we avoid any confusion that might arise in infinite-horizon models, between transversality conditions as conditions of optimality and transversality

[3] The literature has studied at length issues of stability of inflation over time, which arise entirely from the lag that the proxy introduces in the backward-looking Taylor rule. References to that enormous literature and a convincing opinion on the matter can be found in Cochrane (2005a, 2011). In our model, the issue of stability over time does not arise.

[4] The price level at the terminal date is endogenous, like at any date. It is always finite.

conditions as conditions of solvency.[5]

Finally, we depart from most of the recent literature in Monetary Economics in the way we compute and analyze the equilibrium. We do not resort to the customary technique of approximation (by linearization or Taylor expansion) around the deterministic steady state.[6] We handle explicitly the non linearities of the model and obtain an exact solution. We even sometimes discover two viable equilibria. We are able to do that because we reach explicit solutions for the aggregate-demand curve (inclusive of the policy rule) and, in this way, can enumerate and locate solution points exactly.

The empirical estimation of the money-demand curve has become a harder and harder exercise to perform, so much so that, in recent years, many central bankers have stopped paying attention to monetary aggregates and focused exclusively on realized inflation and interest rates. The difficulty, of course, is that money demand and money supply shift simultaneously so that there is an identification problem. We adopt successively two specifications of household monetary behavior, which are standard.[7] The first is the "cashless economy" of Woodford (2003). our second specification will be the "square-root" model of money demand developed some sixty years ago by Allais (1947, pages 238-241), Baumol (1952) and Tobin (1956). In that simple, inventory-theoretic model, households incur a fixed cost every time they go to the bank to turn securities into cash. They regulate their stock of money to minimize the average cost so incurred while making sure that they can always have enough money to meet a fixed, exogenous flow of consumption needs.[8]

As mentioned, the closest antecedents to the present paper are the articles by Benhabib, Schmitt-Grohe and Uribe (2001a, 2001b, 2002), Nakajima and Polemarchaskis (2005) and Magill and Quinzii (2009, 2012). They study issues of indeterminacy (unrelateded to issues of stability) of the price level and of the rate inflation, and their potential solutions by means of Taylor-like intervention rules. The same issues arise here, with the clarity afforded by the finite horizon and with an emphasis on their consequences for the stock market.

[5] See Michel (1982). The reader will observe that, in any case, transversality conditions are stated as limits taken as the terminal date is postponed indefinitely. The model will not be suited to discuss the management of financial-market bubbles by central banks. Furthermore, when we resort to numerical work based on backward induction, terminal conditions are needed as equations that apply to the values of variables, not to their limits.

[6] Recently, some authors have superimposed on the policy rule a zero lower bound on the nominal rate of interest. They were thus lead to worry about non linearities and multiple solutions of the resulting system of equations (which previously was linearized without a qualm). See Fernandez-Villaverde et al. (2012), Mertens and Ravn (2012), Aruoba and Schorfeide (2013), Christiano and Eichenbaum (2013) and Braun et al. (2013).

[7] See Tin (2000).

[8] While several attempts have been made at developing general-equilibrium versions of the Baumol-Tobin model (see, e.g., Romer (1986), Smith (1986), Heathcote (1998), Schwartz (2006), Leo (2006), Bai (2005), Silva (2011) etc..), most of them tend to simplify the model by postulating, e.g., an overlapping-generation model, for the sole purpose of cutting down to size the dynamic program to be solved. Danthine and Donaldson (1986) assume a money demand resulting from money in the utility function and an exogenous supply of money. In that context, they establish the conditions under which stock returns and inflation are negatively correlated.

Some more recent theoretical contributions that deal with asset prices in New Keynesian settings include Nístico (2005), De Paoli, Scott and Weeken (2007), Milani (2008), Li and Palomino (2009), Wei (2009), Castelnuovo and Nístico (2010) and Challe and Giannitsarou (2014), who focus on the response of the stock market to a monetary policy shock, whereas we focus on a productivity shock. And Rudebusch and Swanson (2008) provide a calibration and apply it to bond prices.

The article is organized as follows. In Section 1, we examine the extant empirical evidence concerning the relation between stock returns, bond returns and inflation. In Section 2, we set up a purely financial economy in which income is given exogenously. In Section 3, we add to the financial side of the economy a productive sector in which oligopolistic firms can set prices in a fully flexible way. In Section 4, the productive sector functions along the lines of the New Keynesian model with Calvo pricing. Section 5 provides the main result of the paper as it derives the connection between stock returns and inflation; extensive simulations are performed and the degree to which the model matches the evidence of Section 1 is discussed. In Section 6, we add a demand for cash in the form of a Baumol-Tobin inventory demand and perform new simulations. Finally, in Section 7 we consider the pricing of bonds and discuss the 'Fed model' of price comparison between bonds and stocks, as spelled out in Asness (2003).

1 A brief survey of the empirical evidence

The empirical evidence most cogently related to the present paper pertains to the relations between stock returns and inflation. Lintner (1975), Bodie (1976), Jaffee and Mandelker (1976), Nelson (1976), Fama and Schwert (1977), Fama (1981), Gultekin (1983), Boudoukh and Richardson (1993), Goto and Valkanov (2000) all document a negative correlation between nominal stock returns and inflation at monthly frequency.

To explain the negative correlation, Fama (1981) suggested a "proxy hypothesis" also echoed in the frontispiece of our article. When money demand is stable and money supply is fixed so that no monetary effect is at play, a positive real shock both increases real stock returns and reduces inflation. The negative correlation is then just due to the existence of real shocks. The model to be outlined below does not satisfy Fama's money-supply assumptions; yet the negative correlation we find will also be attributable to real productivity shocks. To the opposite of Fama, Geske and Roll (1983) suggested that the negative response is due to counter-cyclical monetary policy and the monetization of government debt.

Boudoukh and Richardson (BR), whose dataset covers close to two hundred years of annual data, introduce an important distinction between the *ex ante* and the *ex post* forms of the correlation of stock returns with inflation. To capture the *ex post* correlation, BR simply regress one-year holding-period realized stock returns on one-year realized inflation. They do the same for five-year holding-period realized stock returns and five-year realized inflation. In both cases the slope coefficient is found to be significantly positive but it is many times larger for the five-year data.[9]

The *ex ante* relation, otherwise called the "Fisher" hypothesis (here applied to stocks as opposed to bonds or Treasury Bills), relates conditionally expected nominal stock returns to conditionally expected inflation. Under the null hypothesis, the regression slope is expected to be equal to 1, reflecting a constant real rate of return. When anticipating inflation, agents have available an information set, which the econometrician treats as instrumental variables. BR use past inflation and past interest rates as instrumental variables. They do not reject the null hypothesis on five-year data but reject it on one-year data.

Katz and Lustig (2014) using a panel of countries confirm that stock markets are slow to incorporate news about future inflation, while bond markets are not. Gorodnichenko and Weber (2016) show empirically that, after monetary policy announcements, the conditional volatility of stock market returns rises more for firms with stickier prices than for firms with more flexible prices and that sticky prices are, indeed, costly for firms.

2 Building up the financial (or aggregate-demand) side of the model

We begin our investigation with an economy in which economic agents need no money to transact and in which prices of goods and services are fully flexible.

We consider a financial market populated with one (or a continuum of identical) household(s), for which we use a subscript 1, and one central bank, subscripted 2, and a set of exogenous time sequences of individual income received by households $\{y_t \in \mathbb{R}_{++}; t = 0, ...T\}$, which are placed on a tree or lattice. These are received by the households only. For simplicity, we consider a binomial tree so that a given node at time t is followed by two nodes at time $t+1$ at which the two values of income are denoted $\{y_{t+1,u}, y_{t+1,d}\}$. The transition probabilities are equal to 1/2. Notice that the tree accommodates the exogenous state variables only.

In the financial market, there are several securities with at least one nominally riskless security, viz. a one-period nominal bond. The household trades all securities to maximize some lifetime utility. The central bank only trades the one-period nominal bond in a mechanical way described by a policy rule:

Taylor rule at time t

$$1 + i_t = (1 + \bar{\imath}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u} + \frac{1}{2}P_{t+1,d}}{P_t}}{1 + \bar{\pi}} \right)^{\phi} ; \phi \geq 0; \phi \neq 1 \tag{1}$$

[9] Goto and Valkanov (2002) and Hagmann and Lenz (2005), using a vector autoregression, show an attenuation of the negative relationship following the Volcker reform of monetary policy.

Note that the Taylor rule aims to set expected inflation. It does not respond to realized inflation, so that it should have little effect on its conditional volatility.

The government's primary surplus (taxes in excess of expenditures) is denoted s_t in real terms, S_t in nominal terms. The number of units (measured by the nominal amount of the future payoff) of the one-period bond with which the private sector *exits* time t is denoted $\theta_{1,t}$ and its exiting financial wealth $F_{1,t} \triangleq \theta_{1,t}/\left(1+i_t\right)$ is the present value of the nominally riskless bond holdings. We handle the stock market separately as the central bank does not trade it anyway.

We assume that the utility function of the private sector is time-additive and isoelastic. Let the relative risk aversion of the household be $1-\gamma$ and their impatience factor be $\rho < 1$. The private sector (agent carrying a subscript 1) maximizes:

$$\sup_{\{c,\theta_1\}} \mathbb{E}_0 \sum_{t=0}^{T} u\left(c_t, t\right)$$

subject to:

- terminal conditions:
$$\theta_{1,T} = 0, \tag{2}$$
- a sequence of flow budget constraints:
$$P_t \times c_t + \frac{\theta_{1,t}}{1+i_t} + s_t \times P_t = \theta_{1,t-1} + P_t \times y_t \tag{3}$$
- and given initial holdings:
$$\theta_{1,-1} = \bar{\theta}_1 \tag{4}$$

The initial condition at $t = 0$ is given in terms of a *nominal* outstanding claim $\bar{\theta}_1 = -\bar{\theta}_2$ of the public on the government. Please, bear in mind that $\theta_{2,t}$ is a negative number, except in very unusual and temporary fiscal situations.

The government (agent carrying a subscript 2) acts mechanically according to the constraints:

$$\frac{\theta_{2,t}}{1+i_t} = \theta_{2,t-1} + s_t \times P_t; \theta_{2,T,j} = 0; j = u, d \tag{5}$$

and to the Taylor rule (1) with initial holdings:

$$\theta_{2,-1} = \bar{\theta}_2$$

and terminal condition:

$$\theta_{2,T} = 0$$

2.1 Non Ricardian fiscal policy and exogenous supply

Citing Nakajima and Polemarchakis (2005), "a fiscal policy is called 'Ricardian' if it guarantees that the public debt vanishes at each terminal node *for all possible, equilibrium or non-equilibrium, values of price levels and other endogenous variables*" [Emphasis added]. In that case, the fiscal surplus cannot be exogenous throughout. Nakajima and Polemarchakis (2005) demonstrates that, as long as fiscal policy is Ricardian in a cashless economy, the value of government debt is indeterminate.[10]

For that reason, in the balance of this paper, we maintain the assumption of non Ricardian fiscal policy. Therefore, let government surplus s_t be exogenously fixed in real terms. As explained in the introduction, the government's debt is managed mechanically according to a Taylor rule (1), which aims to anchor inflationary expectations.[11]

Definition 1 *An equilibrium is defined as a joint process for the allocation of consumption c_t, the price level P_t, the amount of government bonds outstanding $\theta_{2,t}$ and the nominal rate of interest i_t such that the supremum of the private sector's objective function (30) is reached for all t, the government abides by its period budget constraints (5) and follows the mechanical rule (1), and the market-clearing conditions:*

$$\theta_{1,t} + \theta_{2,t} = 0 \tag{6}$$

are also satisfied with probability 1 *at all times* $t = 0, ...T$.

2.1.1 Equation system

It is shown in Appendix A that a recursive (backward-induction) equilibrium can be obtained by solving, at each node (the exogenous state variable here being c_t) and each point of the grid for the endogenous state variable (here

[10] This is in conformity with Woodford (2003, page 125) and Cochrane (2011). And this indeterminacy induces an indeterminacy of the entire future path of inflation. When later we introduce a production side of the economy, the indeterminacy would also be physical.

[11] It is asserted in Canzoneri et al. (2011) that the backward looking Taylor rule based on realized inflation is incompatible with non Ricardian fiscal policy. With a forward looking Taylor rule involving expected inflation, there is no incompatibility as we show now.

being P_t), the following system of equations:

$$
\begin{gathered}
\text{Flow budget constraints of private sector at time } t+1 \\
P_{t+1,u} \times c_{t+1,u} + F_{1,t+1,u} + s_{t+1,u} \times P_{t+1,u} = \theta_{1,t} + P_{t+1,u} \times y_{t+1,u}; F_{1,T,u} = 0 \\
P_{t+1,d} \times c_{t+1,d} + F_{1,t+1,d} + s_{t+1,d} \times P_{t+1,d} = \theta_{1,t} + P_{t+1,d} \times y_{t+1,d}; F_{1,T,d} = 0 \\
\text{Flow budget constraints of government at time } t+1 \\
F_{2,t+1,u} = \theta_{2,t} + s_{t+1,u} \times P_{t+1,u}; F_{2,T,u} = 0 \\
F_{2,t+1,d} = \theta_{2,t} + s_{t+1,d} \times P_{t+1,d}; F_{2,T,d} = 0 \\
\text{Portfolio-choice or Euler or Fisher condition at time } t \qquad (7) \\
\frac{1}{1+i_t}\frac{1}{P_t} = \rho \frac{\frac{1}{2}\left(c_{t+1,u}\right)^{\gamma-1}\frac{1}{P_{t+1,u}} + \frac{1}{2}\left(c_{t+1,d}\right)^{\gamma-1}\frac{1}{P_{t+1,d}}}{\left(c_t\right)^{\gamma-1}} \\
\text{Taylor rule at time } t \\
1+i_t = (1+\bar{\imath}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u}+\frac{1}{2}P_{t+1,d}}{P_t}}{1+\bar{\pi}}\right)^{\phi} \\
\text{Market clearing at time } t \\
\theta_{1,t} + \theta_{2,t} = 0
\end{gathered}
$$

The functions carried backward in the backward-induction procedure are $F_{1,t}$ $(= -F_{2,t})$:

$$F_{1,t} \triangleq \frac{\theta_{1,t}}{1+i_t}.$$

The unknowns are
$\{i_t, c_{t+1,u}, c_{t+1,d}, \theta_{1,t}, \theta_{2,t}, P_{t+1,u}, P_{t+1,d}\}$. The only endogenous state variable is the current price level P_t, which is determined at time zero from the outstanding nominal amount of government debt as in the Fiscal Theory of the Price Level.[12] At time zero, the *initial condition* to be solved for the unknown initial price P_0 is

$$f_{2,0} \times P_0 = \theta_{2,-1} + s_0 \times P_0 \qquad (8)$$

where $\theta_{2,-1}$ is a given (negative) amount of nominal claim outstanding and s_0 a given time-0 surplus.

Since the government does not trade the *equity*, and private agents are homogeneous, it is not traded at all. If the equity security is defined – for the time being – as paying the total income, its price x_t is virtual and equal to

$$
\begin{aligned}
x_t &= \rho \frac{\frac{1}{2}\left(c_{t+1,u}\right)^{\gamma-1} \times \left(y_{t+1,u} + x_{t+1,u}\right) + \frac{1}{2}\left(c_{t+1,d}\right)^{\gamma-1} \times \left(y_{t+1,d} + x_{t+1,d}\right)}{\left(y_t\right)^{\gamma-1}}; \\
x_T &= 0 \qquad (9)
\end{aligned}
$$

[12] Sims (1994) introduced the Fiscal theory of the price level. See Cochrane (2005a) and Niepelt (2004).

The formula for the real price x_t of the stockmarket (9) owes nothing to the price level but the real rate of return on it is conditionally correlated with inflation since $y_{t+1}, s_{t+1}, f_{2,t+1}$ are correlated with each other.

2.1.2 Analytical solution

There exists an analytical solution for which

$$c_{t+1,u} = y_{t+1,u}; c_{t+1,d} = y_{t+1,d}$$

We now derive that solution.

Government debt is nominal and can be priced by means of the Fisher equation, which means that the financial wealth of the government can be obtained by the following backward induction:

$$\frac{F_{2,t}}{P_t} \triangleq \frac{1}{P_t}\frac{\theta_{2,t}}{1+i_t} \tag{10}$$
$$= \rho \frac{\frac{1}{2}\left(c_{t+1,u}\right)^{\gamma-1}\left(-s_{t+1,u}+\frac{F_{2,t+1,u}}{P_{t+1,u}}\right)+\frac{1}{2}\left(c_{t+1,d}\right)^{\gamma-1}\left(-s_{t+1,d}+\frac{F_{2,t+1,d}}{P_{t+1,d}}\right)}{\left(y_t\right)^{\gamma-1}}$$

Assume homogeneity with respect to the price level (with notation: $\theta_{1,t} \equiv \vartheta_{1,t} \times P_t; \theta_{2,t} \equiv \vartheta_{2,t} \times P_t; F_{1,t+1,u} \equiv f_{1,t+1,u} \times P_{t+1,u}; F_{2,t} \equiv f_{2,t} \times P_t; F_{2,t+1,u} \equiv f_{2,t+1,u} \times P_{t+1,u}$) and assume: $f_{1,t+1,u} = -f_{2,t+1,u}, f_{1,t+1,d} = -f_{2,t+1,d}$. The system of equations becomes

Flow budget constraints of private sector

$$-f_{2,t+1,u} \times P_{t+1,u} + s_{t+1,u} \times P_{t+1,u} = \vartheta_{1,t} \times P_t$$
$$-f_{2,t+1,d} \times P_{t+1,d} + s_{t+1,d} \times P_{t+1,d} = \vartheta_{1,t} \times P_t$$

Flow budget constraints of government

$$f_{2,t+1,u} \times P_{t+1,u} = \vartheta_{2,t} \times P_t + s_{t+1,u} \times P_{t+1,u} \tag{11}$$
$$f_{2,t+1,d} \times P_{t+1,d} = \vartheta_{2,t} \times P_t + s_{t+1,d} \times P_{t+1,d} \tag{12}$$

Portfolio-choice or Euler or Fisher condition

$$\frac{1}{1+i_t}\frac{1}{P_t} = \rho \frac{\frac{1}{2}\left(y_{t+1,u}\right)^{\gamma-1}\frac{1}{P_{t+1,u}}+\frac{1}{2}\left(y_{t+1,d}\right)^{\gamma-1}\frac{1}{P_{t+1,d}}}{\left(y_t\right)^{\gamma-1}}$$

Taylor rule

$$1+i_t = (1+\bar{\imath}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u}+\frac{1}{2}P_{t+1,d}}{P_t}}{1+\bar{\pi}}\right)^{\phi}$$

Market clearing

$$\vartheta_{1,t} + \vartheta_{2,t} = 0$$

Government debt: From (10), the backward dynamics of real government financial liabilities are provided by:

$$\begin{aligned} f_{2,t} &= \rho \frac{\frac{1}{2}(y_{t+1,u})^{\gamma-1}(-s_{t+1,u}+f_{2,t+1,u})+\frac{1}{2}(y_{t+1,d})^{\gamma-1}(-s_{t+1,d}+f_{2,t+1,d})}{(y_t)^{\gamma-1}}; \\ f_{2,T} &= 0 \end{aligned} \tag{13}$$

The real discounted value $f_{2,t}$ depends only on future output and future surpluses. It does not depend on interest-rate policy. But the real face value $\vartheta_{2,t}$, which is the government's equilibrium portfolio choice, depends on the nominal rate of interest, which we now determine.

Interest rate and inflation: Solving for inflation from the government flow budget constraints (11), (12):

$$\begin{aligned} \frac{P_{t+1,u}}{P_t} &= \frac{\vartheta_{2,t}}{-s_{t+1,u}+f_{2,t+1,u}} \\ \frac{P_{t+1,d}}{P_t} &= \frac{\vartheta_{2,t}}{-s_{t+1,d}+f_{2,t+1,d}} \end{aligned}$$

so that the realized rates of inflation are:

$$\begin{aligned} \frac{P_{t+1,u}}{P_t} &= \frac{f_{2,t}\times(1+i_t)}{-s_{t+1,u}+f_{2,t+1,u}} \\ \frac{P_{t+1,d}}{P_t} &= \frac{f_{2,t}\times(1+i_t)}{-s_{t+1,d}+f_{2,t+1,d}} \end{aligned}$$

These relations between the rate of inflation and the nominal rate of interest are commonly known as the "aggregate-demand" schedules.[13]

We now merge them with the policy rule. Substituting into the Taylor rule:

$$1+i_t = (1+\bar{\imath})\times\left(\frac{\frac{1}{2}\frac{\vartheta_{2,t}}{-s_{t+1,u}+f_{2,t+1,u}}+\frac{1}{2}\frac{\vartheta_{2,t}}{-s_{t+1,d}+f_{2,t+1,d}}}{1+\bar{\pi}}\right)^{\phi}$$

so that (using $\vartheta_{2,t}/(1+i_t)=f_{2,t}$):

$$1+i_t = (1+\bar{\imath})^{\frac{1}{1-\phi}}\times\left(f_{2,t}\frac{\frac{1}{2}\frac{1}{-s_{t+1,u}+f_{2,t+1,u}}+\frac{1}{2}\frac{1}{-s_{t+1,d}+f_{2,t+1,d}}}{1+\bar{\pi}}\right)^{\frac{\phi}{1-\phi}} \tag{14}$$

The nominal rate of interest depends on future fiscal surpluses and output, as well as on the parameters of the Taylor rule.

Finally, since (13) provides a unique value for the time-0 present value of the government debt, and since $\theta_{2,-1}$ is a given (negative) amount of nominal claim outstanding and s_0 a given time-0 surplus, the solution of the initial condition (8) for P_0 is unique. Cochrane (2011, page 579) says that we have determinacy

[13] The next two sections derive the "aggregate-supply" schedule. In the present section aggregate supply is exogenous and completely inelastic.

in this case and, indeed, we do, irrespective of the value of the Taylor parameter so long as $\phi \neq 1$.

The solution of the system relates the two levels of future inflation $(P_{t+1,u}/P_t, P_{t+1,d}/P_t)$ to calendar time t, to the two levels of future real government debt $(-s_{t+1,u} + f_{2,t+1,u}, -s_{t+1,d} + f_{2,t+1,d})$ and to the current level of real government debt $f_{2,t}$. We call $f_{2,t}/\left(-s_{t+1} + f_{2,t+1}\right)$ the "ex post *inverse* real gross rates of return on government debt". It is also the ex post *inverse* real gross rates of return on any *nominally* riskless debt.

Proposition 2 *Under isoelastic utility, the ex post levels of inflation are*

- *increasing functions of the ex post* inverse *real gross rates of return on nominally riskless debt*
- *increasing functions of the ex ante nominal gross rate of interest, which is itself*
  - *an increasing (decreasing) function of the expected* inverse *real gross rate of return on nominally riskless debt if $\phi < 1$ ($\phi > 1$).*

In total, the higher ex post inverse real gross rates of return on government debt has a double effect, one direct and increasing and one indirect because it affects the expected value of the inverse real gross rates of return. The second effect is ambiguous, its sign depending on whether ϕ is smaller or greater then 1. When $\phi < 1$, the direction of the effect is clear: a higher real rate of return on government implies a lower rate of inflation.

2.1.3 Special case

In case growth is stochastic and identically and independently distributed (IID) over time:

$$\frac{y_{t+1,u}}{y_t} = 1 + u; \frac{y_{t+1,d}}{y_t} = 1 + d; u \text{ ("up")} > d \text{ ("down")}$$

there is scale invariance in the sense that the quantity $f_{2,t}$ does not depend on the level of income y_t at time t once the surplus process $\{s_t\}$ is given.

If, however, the exogenous surplus is specified to be at all time and in all states proportional to income, $s_t = \tau \times y_t$, where τ can interpreted as a constant tax rate, then the real discounted value of government debt $f_{2,t}$ is proportional to the level of income y_t at time t :

$$f_{2,t} = \hat{f}_{2,t} \times y_t$$

where $\hat{f}_{2,t}$ declines deterministically as one approaches the terminal date. Indeed:

$$\frac{\hat{f}_{2,t}}{-\tau + \hat{f}_{2,t+1}} = \rho \times \left[\frac{1}{2}(1+u)^{\gamma} + \frac{1}{2}(1+d)^{\gamma}\right]; \hat{f}_{2,T} = 0$$

The realized inverse real rates of return on government debt are:

$$\frac{f_{2,t}}{-s_{t+1,u}+f_{2,t+1,u}} = \rho\times\frac{\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}}{1+u} \text{ on an "up" move}$$
$$\frac{f_{2,t}}{-s_{t+1,d}+f_{2,t+1,d}} = \rho\times\frac{\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}}{1+d} \text{ on a "down" move}$$

The realized rates of inflation are:

$$\frac{P_{t+1,u}}{P_t} = \rho\times\frac{\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}}{1+u}\times(1+i_t)$$
$$\frac{P_{t+1,d}}{P_t} = \rho\times\frac{\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}}{1+d}\times(1+i_t)$$

independent of the tax rate. Inflation is lower when output is higher, like in the traditional quantity theory of money but for completely different, in this case fiscal, reasons. The quantity

$$\rho\times\left[\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}\right]\left(\frac{1}{2}\frac{1}{1+u}+\frac{1}{2}\frac{1}{1+d}\right)$$

can be viewed as the expected inverse gross real rate of interest on nominally riskless claims, which is not equal to the inverse gross rate on really riskless claims ($\rho\times\left[(1+u)^{\gamma-1}+(1+d)^{\gamma-1}\right]/2$).

The nominal rate of interest is constant:

$$1+i_t = \left(\frac{1+\bar{\imath}}{(1+\bar{\pi})^{\phi}}\right)^{\frac{1}{1-\phi}} \times\left\{\rho\times\left[\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}\right]\left(\frac{1}{2}\frac{1}{1+u}+\frac{1}{2}\frac{1}{1+d}\right)\right\}^{\frac{\phi}{1-\phi}}$$

Figure 1 contains illustrations for the special case of IID growth and surplus calculated from a constant tax rate. The relations are shown at time $t+1$ for the two cases of an "up" and a "down" output shock for a fixed level of output at time t. The prices set by these aggregate demand curves (inclusive of policy rule) are increasing functions of output when $\phi>1$ and decreasing functions when $\phi<1$.

The stock-market price is:

$$x_t = \hat{x}_t\times y_t; x_T=0$$
$$\hat{x}_t = \rho\times\left[\frac{1}{2}(1+u)^{\gamma}(1+\hat{x}_{t+1,u})+\frac{1}{2}(1+d)^{\gamma}(1+\hat{x}_{t+1,d})\right]$$

where $\hat{x}_t$ is deterministic:

$$\hat{x}_t=\rho\times\left[\frac{1}{2}(1+u)^{\gamma}+\frac{1}{2}(1+d)^{\gamma}\right]\times(1+\hat{x}_{t+1});\hat{x}_T=0$$

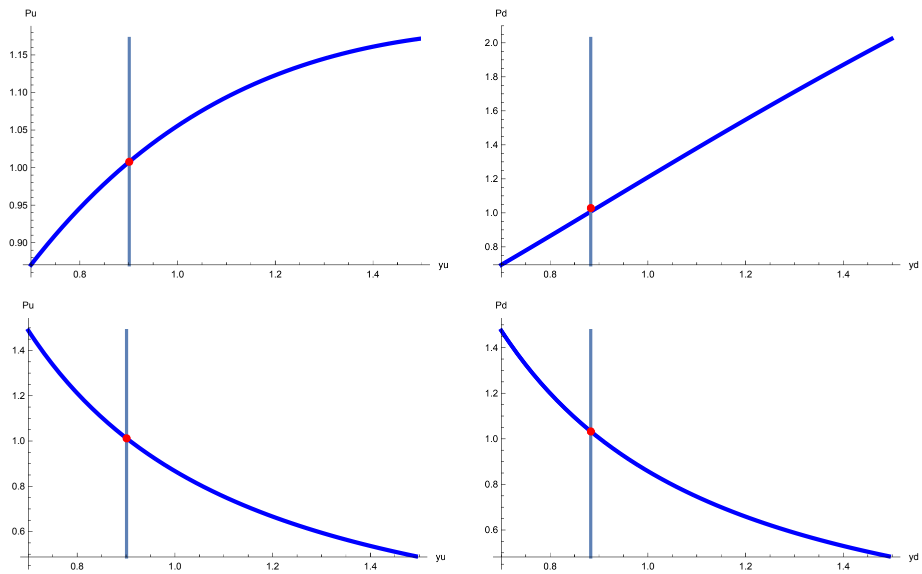

Figure 1: **Aggregate-demand curve (inclusive of policy rule) with fixed outputs, $\phi = 1.5$ (top panels) and $\phi = 0.5$ (bottom panels)**. In each row, the left-hand panel shows $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the fixed value of $y_{t+1,d}$. The fixed value of $y_{t+1,u}$ is shown as a vertical line. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the fixed value of $y_{t+1,u}$. The fixed value of $y_{t+1,d}$ is shown as a vertical line. Parameter values are as in Table 1. $T = 6$. The current price level is set at 1. The current level of output y_t is set at 0.9011.

Parameter	Value
ρ	0.99
$\bar{\pi}$	$2\%/year$
$\bar{\imath}$	$1/\rho - 1 + \bar{\pi}$
$1-\gamma$	1
tax rate τ	1/3
σ	4
η	2
volatility of z growth	$1\%/year$
expected value of z growth	0
ω	0.6

Table 1: Parameter values for the numerical illustration; one-year periods

Not surprisingly, there exists a systematic relation between the real stock market price per unit of output $\hat{x}_t$ (the price-dividend ratio) and the real discounted value of government debt per unit of output $\hat{f}_{2,t}$. The price-dividend ratio $\hat{x}$ is $1/\tau$ times larger than government debt per unit.

$$-\frac{\hat{f}_{2,t}}{\tau} = \hat{x}_t$$

Over time, they both decline deterministically. The real rates of return on the stock market are:

$$-1 + \frac{1+u}{\rho \times \left[\frac{1}{2}(1+u)^{\gamma} + \frac{1}{2}(1+d)^{\gamma}\right]} \text{ on an "up" node}$$

$$-1 + \frac{1+d}{\rho \times \left[\frac{1}{2}(1+u)^{\gamma} + \frac{1}{2}(1+d)^{\gamma}\right]} \text{ on a "down" node}$$

On an up node, inflation is lower than on a down node while the real stock market return is higher but that is just a "proxy" result of a common cause, namely the output shock, which acts both on the stock market and on tax collection. The real rate of return on equity being low when inflation is high, it is negatively correlated with the rate of inflation and the stock market is not a one-for-one hedge against inflation. In fact, for their product, which is the realized gross *nominal* rate of return on stocks, we have:

Proposition 3 *Under the IID assumption of the special case, the realized gross nominal rate of return on stocks is equal to the gross nominal interest rate, which is constant.*

In later sections, we introduce additional features that will produce more realistic outcomes. More importantly, these features will explain the fact that the link between inflation and nominal stock returns varies depending on the length of the holding period.

3 Endogenous, flexible-price aggregate supply

The model we have built in Section 2 constitutes the combination of the policy rule with the aggregate-demand or "IS" side of the economy. The solution we calculated is complete when income or output is exogenous. We now introduce firms and endogenize output. The shocks to output will now derive from productivity shocks. In this section, firms are assumed to set their prices will full flexibility, this being only a transition to the next section where prices will be sticky.

We now develop the aggregate-supply side, which endogenizes total income, productivity z being now the exogenous state variable. In this section, we assume that firms are free to adjust their prices. We use that part of the model to obtain future (time-$t+1$) output.

Households: There exists a continuum $v \in [0,1]$ of differentiated varieties of the good.[14] The argument c_t of the households' utility is a composite defined as

$$c_t \triangleq \left(\int_0^1 c_{v,t}^{\frac{\sigma-1}{\sigma}} dv \right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution between the separate varieties. As a result, their demand for each separate variety v is

$$c_{v,t} = \left(\frac{P_{v,t}}{P_t} \right)^{-\sigma} c_t$$

where $P_{v,t}$ is the nominal price of variety v and P_t is the general price index, which is defined generally as

$$P_t \triangleq \left(\int_0^1 P_{v,t}^{1-\sigma} dv \right)^{\frac{1}{1-\sigma}} \tag{15}$$

but will be particularized below. In addition, the utility function of households now contains a separate, additive term for the dis-utility of labor. The full utility function that households optimize is

$$\sup_{\{c,l,\theta_1\}} \mathbb{E}_0 \sum_{t=0}^{T} u(c_t, t) - \hat{u}(l_t, t)$$

subject to terminal conditions (2), a sequence of flow budget constraints:

$$P_t \times c_t + \frac{\theta_{1,t}}{1+i_t} + \theta_{X,t} \times P_t \times x_t + s_t \times P_t = \theta_{1,t-1} + \theta_{X,t-1} \times P_t \times (\delta_t + x_t) + W_t \times l_t$$

and given initial holdings:

$$\begin{aligned} \theta_{1,-1} &= \bar{\theta}_1 \\ \theta_{X,-1} &= 1 \end{aligned}$$

[14] Here, we follow Chapter 8 in Walsh (2010) and Challe (2005).

where W_t is the nominal wage rate, l_t the number of hours worked, $\theta_{X,t}$ equity holdings, x_t the real price of equity and δ_t real dividends distributed. Since households alone hold the stock, it will be the case at equilibrium that $\theta_{X,t} = \theta_{X,t-1} = 1$. We have in mind, however, that the first-order condition for equity holdings will serve to price the equity.

We assume an isoelastic dis-utility of work: $\rho^t \times l_t^\eta / \eta$. The households' first-order condition for hours worked is obviously

$$\frac{l_t^{\eta-1}}{c_t^{\gamma-1}} = \frac{W_t}{P_t} \tag{16}$$

Firms: The production function for variety v of the good is

$$y_{v,t} = z_t \times l_{v,t}$$

where z_t is a productivity shock, the same for all firms and $l_{v,t}$ is the amount of labor utilized for the production of good v. When choosing to hire labor, firm v minimizes the real cost of producing any given amount $l_{v,t}$ of variety v:[15]

$$\inf_{l_{v,t}} \frac{W_t \times l_{v,t}}{P_t}$$

subject to

$$\begin{aligned} z_t \times l_{v,t} &= y_{v,t} \\ l_{v,t} &\geq 0 \end{aligned} \tag{17}$$

We get

$$l_{v,t} = \begin{cases} +\infty \text{ if } \frac{W_t}{P_t} < \varphi_t \times z_t \\ 0 \text{ if } \frac{W_t}{P_t} > \varphi_t \times z_t \end{cases}$$

where φ_t is the Lagrange multiplier of the constraint (17), to be chosen as usual in such a way that the constraint is satisfied. For the hired labor to be finite, we must have

$$\varphi_t = \frac{W_t}{z_t \times P_t} \tag{18}$$

and we interpret φ_t as the real marginal cost of labor.

Firms are free to adjust their prices at will. They maximize their profits by setting a mark up and an optimal price P^* related to the price-elasticity of demand:

$$\frac{P_t^*}{P_t} = \frac{\sigma}{\sigma - 1}\varphi_t$$

They produce $(P_t^*/P_t)^{-\sigma} \times y_t$. Total labor employed is:

$$\left(\frac{P_t^*}{P_t}\right)^{-\sigma} \times \frac{y_t}{z_t}$$

[15] The overall objective of the firm is to maximize its market value $x_{j,t}$ where $x_{j,t}$ is as in (24) and (25) below. The equity price is set by households in the stock market. That is why the price index P_t appears in the hiring decision.

Letting l_t stand for the labor supplied by households, the clearing of the labor market requires:

$$l_t = \left(\frac{P_t^*}{P_t}\right)^{-\sigma} \times \frac{y_t}{z_t} \tag{19}$$

By Walras' law, the equilibrium in the financial market and the equilibrium in the labor market imply the equilibrium in the goods market: $c_t = y_t$.

Equilibrium: Since all the firms behave the same way, (15) implies that $P_t = P_t^*$. Equations (18), (16) and (19) imply that the the flexible-price level of output is:

$$y_t = \left(\frac{\sigma - 1}{\sigma} z_t^\eta\right)^{\frac{1}{\eta-\gamma}}$$

and that the price is indeterminate. The determination of the price level is then left entirely to the aggregate demand side (inclusive of the policy rule) exactly as in Section 2.

Special case: The special IID case described in Section 2.1.3 can be recast in terms of productivity shocks. The resulting equilibrium diagrams remain identical to Figure 1, reinterpreted as showing *endogenous* values of the output.[16]

Proposition 4 *Under the assumptions of the IID special case, it remains true under flexible prices that the gross nominal rate of return on stocks, is equal to the gross nominal interest rate, which is constant.*

In the next section, we introduce sticky prices. They will explain the main fact that we are trying to understand, i.e., that the link between inflation and nominal stock returns varies depending on the length of the holding period.

4 Endogenous sticky-price aggregate supply

We now develop in standard New Keynesian fashion (see, for instance, Galí (2008), Walsh (2010) or Challe (2005)), the case in which firms are not free to set their prices, thus generating the Phillips curve, which endogenizes total income. The Phillips curve relates the price level to output or total income contemporaneously. We later shift it to time $t+1$, so that, in our rendition, it will relate the future price level to future income.

Firms: Firms are not free to adjust their prices at will. Instead, as in Calvo (1983), each firm at each point in time has a probability $1-\omega$ of being allowed to adjust its price to an optimal level P_t^* (which will be the same for all firms). By the Law of Large Numbers, a fraction $1-\omega$ do so, so that the price index

[16] In each row, the left-hand panel shows $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the flexible-price value of $y_{t+1,d}$. The flexible-price value of $y_{t+1,u}$ is shown as a vertical line. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the flexible-price value of $y_{t+1,u}$. The flexible-price value of $y_{t+1,d}$ is shown as a vertical line. Additional parameter values are as in Table 1. The current level of output y_t is set at its flexible-price level equal to 0.9011.

particularizes to:[17]

$$P_t \triangleq \left[(1-\omega)\times\left(P_t^*\right)^{1-\sigma}+\omega\times\left(P_{t-1}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{20}$$

Firms maximize their market value on the equity market. With regard to setting its current price $P_{v,t}$, the part of each firm's objective function that depends on it is:[18]

$$\sup_{P_{v,t}}\sum_{i=0}^{T-t}\mathbb{E}_t\left[(\rho\omega)^i\frac{\left(c_{t+i}\right)^{\gamma-1}}{\left(c_t\right)^{\gamma-1}}\left(\frac{P_{v,t}}{P_{t+i}}-\varphi_{t+i}\right)\left(\frac{P_{v,t}}{P_{t+i}}\right)^{-\sigma}y_{t+i}\right]$$

(where: $y_t \triangleq \left(\int_0^1 y_{v,t}^{\frac{\sigma-1}{\sigma}}dv\right)^{\frac{\sigma}{\sigma-1}}$) with a solution $P_{v,t}=P_t^*$ which is:[19]

$$\frac{P_t^*}{P_t}=\frac{\sigma}{\sigma-1}\frac{\mathbb{E}_t\sum_{i=0}^{T-t}(\rho\omega)^i\left(c_{t+i}\right)^{\gamma-1}y_{t+i}\varphi_{t+i}\left(\frac{P_{t+i}}{P_t}\right)^{\sigma}}{\mathbb{E}_t\sum_{i=0}^{T-t}(\rho\omega)^i\left(c_{t+i}\right)^{\gamma-1}y_{t+i}\left(\frac{P_{t+i}}{P_t}\right)^{\sigma-1}} \tag{21}$$

a function of y_t for which the numerator and the denominator will be computed by backward induction. To that aim, we restate Equation (21) in recursive form:

$$\begin{aligned}
\frac{P_t^*}{P_t} &= \frac{\sigma}{\sigma-1}\frac{c_t^{\gamma-1}y_t\varphi_t+A\left(t,y_t\right)}{c_t^{\gamma-1}y_t+B\left(t,y_t\right)} \qquad (22)\\
A\left(t,y_t\right) &\triangleq \mathbb{E}_t\rho\omega\left(\frac{P_{t+1}}{P_t}\right)^{\sigma}\left[\left(c_{t+1}\right)^{\gamma-1}y_{t+1}\varphi_{t+1}+A\left(t+1,y_{t+1}\right)\right]\\
A\left(T,y_T\right) &= 0\\
B\left(t,y_t\right) &\triangleq \mathbb{E}_t\rho\omega\left(\frac{P_{t+1}}{P_t}\right)^{\sigma-1}\left[\left(c_{t+1}\right)^{\gamma-1}y_{t+1}+B\left(t+1,y_{t+1}\right)\right]\\
B\left(T,y_T\right) &= 0
\end{aligned}$$

Equilibrium: As a result of their choice of price, a proportion ω of firms produce $\left(P_{t-1}/P_t\right)^{-\sigma}\times y_t$ on an average and employ $\left(P_{t-1}/P_t\right)^{-\sigma}\times y_t/z_t$ units of labor and a proportion $1-\omega$ of firms produce $\left(P_{t-1}/P_t\right)^{-\sigma}\times y_t$ and employ

[17] This equation should really be:

$$P_t \triangleq \left[\int_0^1(1-\omega)\times\left(P_{v,t}^*\right)^{1-\sigma}+\omega\times\left(P_{v,t-1}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

We are going to find that $P_{v,t}^*$ is the same for all v but that is not true for $P_{v,t-1}$. The index of price dispersion across firms should really be present in the derivations below. We ignore it. For more details on this, see the appendix of Challe and Giannitsarou (2014). We thank Edouard Challe for confirmation.

[18] The overall objective function is the maximization of equity value, which includes additional terms not dependent on $P_{v,t}$. See the value of the stock market (24) and (25) below.

[19] The proof is standard and is reproduced in Appendix B.

$(P_{t-1}/P_t)^{-\sigma} \times y_t/z_t$ units of labor.[20] Total labor employed is:

$$\left[\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_t^*}{P_t}\right)^{-\sigma}\right] \times \frac{y_t}{z_t}$$

Letting l_t stand for the labor supplied by households, the clearing of the labor market requires:

$$l_t = \left[\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_t^*}{P_t}\right)^{-\sigma}\right] \times \frac{y_t}{z_t} \tag{23}$$

Substitution of Equations (18), (16), (23) and (20) into (22) gives the equilibrium Phillips curve $P_t/P_{t-1} = Phill_t(y_t)$ in implicit form:[21]

$$\left(\frac{1-\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{1-\sigma}}{1-\omega}\right)^{\frac{1}{1-\sigma}} = \frac{\sigma}{\sigma-1}\frac{1}{y_t^\gamma + B(t, y_t)}$$

$$\times \left\{ \left(\frac{y_t}{z_t}\right)^\eta \left[\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{-\sigma} + (1-\omega) \times \left(\frac{1-\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{1-\sigma}}{1-\omega}\right)^{-\frac{\sigma}{1-\sigma}}\right]^{\eta-1} + A(t, y_t) \right\}$$

where:

$$A(t, y_t) = \rho\omega\mathbb{E}_t \left(\frac{P_{t+1}}{P_t}\right)^\sigma$$

$$\times \left\{ \left(\frac{y_{t+1}}{z_{t+1}}\right)^\eta \left[\omega \times \left(\frac{P_t}{P_{t+1}}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_{t+1}^*}{P_{t+1}}\right)^{-\sigma}\right]^{\eta-1} + A(t+1, y_{t+1}) \right\}$$

and:

$$B(t, y_t) = \rho\omega\mathbb{E}_t \left(\frac{P_{t+1}}{P_t}\right)^{\sigma-1} \left[y_{t+1}^\gamma + B(t+1, y_{t+1})\right]$$

[20] Because of (20):

$$y_t \equiv \left\{ \omega \times \left[\left(\frac{P_{t-1}}{P_t}\right)^{-\sigma} \times y_t\right]^{\frac{\sigma-1}{\sigma}} + (1-\omega) \times \left[\left(\frac{P_t^*}{P_t}\right)^{-\sigma} \times y_t\right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

i.e., the amounts produced by the two categories of firms add up to y_t.

[21] As we saw in the previous section, in the case of full price flexibility ($\omega = 0$), the Phillips curve is vertical at the flexible-price level of output: $y_t = \left((\sigma-1) z_t^\eta/\sigma\right)^{1/(\eta-\gamma)}$ and, on its own, leaves the price indeterminate.

There exists an approximate, explicit form for the Phillips function, as suggested in Galí (2008) and developed in our Appendix C.

The shapes of the Phillips curves are illustrated in Figure 2 and 3 (to be commented upon below), along with the accompanying aggregate-demand curves re-derived according to Section 2.1.2.

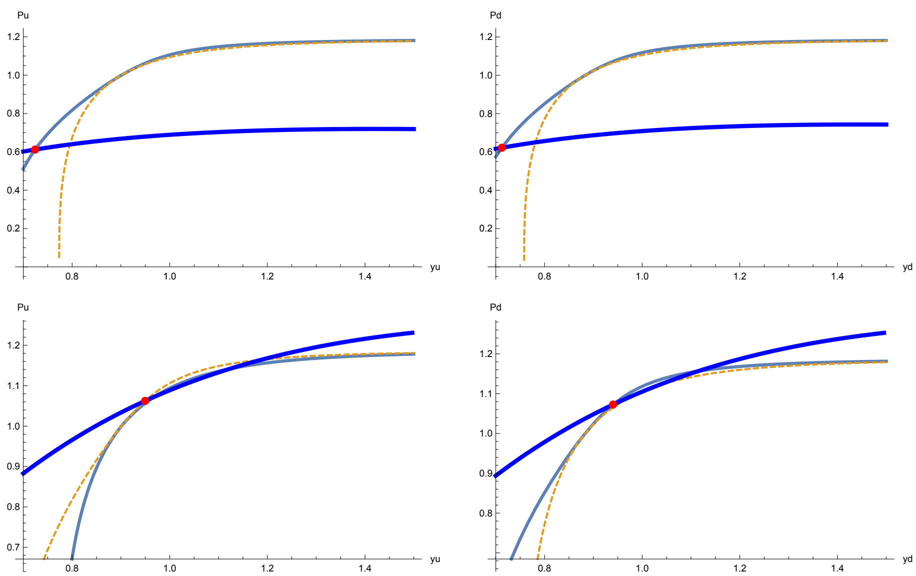

Figure 2: **Aggregate-demand (inclusive of policy rule) and aggregate-supply curves with sticky-price output and $\phi = 1.5$. Top panels: low-output equilibrium. Bottom panels: high-output equilibrium** (leftmost intersection)**.** In each row, left-hand panel: $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the equilibrium sticky-price value of $y_{t+1,d}$. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the equilibrium sticky-price value of $y_{t+1,u}$. The lighter solid line is the Phillips or aggregate-supply curve; the dashed line is the approximate Phillips curve of Appendix C and the darker solid line is the aggregate-demand (inclusive of policy rule) curve. Parameter values are as in Table 1. $T = 6$. The current price level is set at 1. The current level of output y_t is set at 3.5% above the flexible-price level.

The time $t+1$ Phillips curves and the system to be solved: Because the time-t aggregate-demand relations established before relate time-$t+1$ prices to time-$t+1$ output, it is convenient to shift the Phillips curves to time $t+1$. In this way, we are left with a system of four equations in four unknowns: $\{P_{t+1,u}, P_{t+1,d}, y_{t+1,u}, y_{t+1,d}\}$ which must be solved numerically for each node of the tree (each capturing exogenous state variable z_t) and for each value of the endogenous state variable y_t, recursively for $t = T-1, ..., 0$. The current price P_t is also an endogenous state variable but, in the absence of money illusion, it

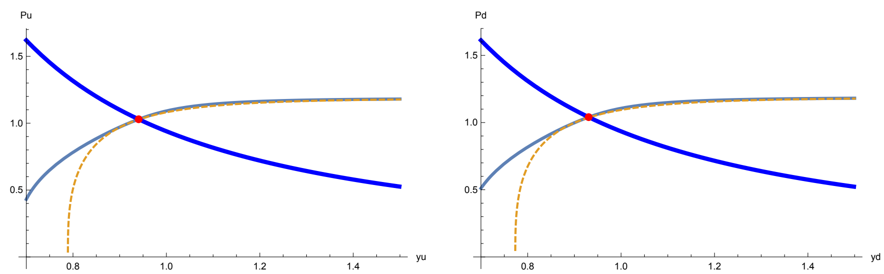

Figure 3: **Aggregate-demand (inclusive of policy rule) and aggregate-supply curves with sticky-price output and $\phi = 0.5$; single equilibrium.** Left-hand panel: $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the equilibrium sticky-price value of $y_{t+1,d}$. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the equilibrium sticky-price value of $y_{t+1,u}$. The lighter solid line is the Phillips or aggregate-supply curve; the dashed line is the approximate Phillips curve of Appendix C and the darker solid line is the aggregate-demand (inclusive of policy rule) curve. $T = 6$. Additonal parameter values are as in Table 1. The current price level is set at 1. The current level of output y_t is set at 3.5% above the flexible-price level.

can be factored out on grounds of homogeneity.[22]

By Walras' law, the equilibrium in the financial market and the equilibrium in the labor marlet imply the equilibrium in the goods market: $c_{t+1,u} = y_{t+1,u}$ and $c_{t+1,d} = y_{t+1,d}$.[23]

The shapes of the aggregate-demand and Phillips curves are such that there may not exist solutions, that there may be multiple solutions and that gradient-based solvers do not find them easily. We may find our way towards one of the solution by starting with the flexible-price solution ($\omega = 0$) and by gradually increasing ω in small increments, or by starting at no price adjustment ($\omega = 1$) and by gradually decreasing ω.

When $\phi > 1$, there are several solutions,[24] two of which are shown in the two panels of Figure 2, for the point in time $t = T - 1$. In such a case, it is impossible to pursue the recursion to earlier points in time. This difficulty would not have even been spotted by the large number of researchers who work not with the exact system of equations but with a system that is linearized around the flexible-price solution.

[22] In addition, when household utility is isoelastic and the production function satisfies the property of constant returns ot scale, a scale-invariance property can be exploited: we need not do the calculation for every node of each point in time t, which differ only in the level of productivity z_t. For the several nodes of time t, the functions that are carried backward (f_1 or f_2, A and B) can be deduced from a single one of them.

[23] In Appendix D, we verify that clearing of the financial market and of the labor market do imply clearing of the goods market.

[24] And, for low enough values of current output y_t there are no solutions.

When $\phi < 1$, the equilibrium is almost surely unique as shown in Figure 3. The reason is that, in that case, as we have seen under Proposition 2, the aggregate-demand functions (inclusive of the policy rule) are decreasing, while the Phillips curve, of course, is increasing. In what follows, we assume that $\phi < 1$ so that we can obtain a unique solution at any point in time.

Initial conditions: At time zero, the initial condition to be solved for the unknown initial price P_0 and the initial income y_0 given the initial productivity z_0 are:

$$\begin{aligned} f_{2,0} \times P_0 &= \theta_{2,-1} + s_0 \times P_0 \\ \frac{P_0}{P_{-1}} &= Phill\left(0, y_0\right) \end{aligned}$$

where the first condition is identical to the initial condition of Section 2, the second one is just the Phillips curve at time 0 and where $\theta_{2,-1}$ is a given (negative) amount of nominal claim outstanding and s_0 a given time-0 surplus. The solution for P_0 is unique as long as the backward recursion provided a unique function $Phill$.

Stationary solution: As mentioned, we solve the system for each node of the tree (each node capturing exogenous state variable z_t) and for each value of the endogenous state variable y_t, recursively for $t = T-1, ..., 0$. With the impatience parameter set at $\rho = 0.99$, the value $T = 270$ years is sufficiently large for functions carried backward to be unchanging by the time we get to time 0. The stationary functions capture the equilibrium of an economy with an horizon that has been increased indefinitely.

5 Stock returns and inflation

We define the aggregate stock security as paying corporate profits (as opposed to paying output, which it was in Section 2)). The real, future profits, assumed to be distributed as dividends, are:[25]

$$\begin{aligned} \delta_{t+1} \triangleq & \left[\omega \times \left(\frac{P_t}{P_{t+1}} - \varphi_{t+1}\right)\left(\frac{P_t}{P_{t+1}}\right)^{-\sigma} \right. \\ & \left. + (1-\omega) \times \left(\frac{P^*_{t+1}}{P_{t+1}} - \varphi_{t+1}\right)\left(\frac{P^*_{t+1}}{P_{t+1}}\right)^{-\sigma}\right] \times y_{t+1} \end{aligned} \tag{24}$$

The value of the stock market, in real terms, current profits not included, is:

$$x_t = \rho \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1} (\delta_{t+1} + x_{t+1})\right] \tag{25}$$

Numerical illustrations will indicate the correlation between the rate of return on the stock market and inflation.

[25] Current profit δ_t differs from one firm to the other, depending on which firm is allowed currently to change its price. For future profits, we ignore current price dispersion, as explained in footnote 17.

5.1 Impulse response to a productivity shock

After solving all the equations of all times in a backward sequence, we use the stationary functions to simulate the economy, drawing at random the event of an "up" or a "down" productivity shock z over 200 time steps of one year each. Ten thousand paths are drawn. For this and the next simulations, we now set the target rate $\bar{\imath}$ to be approximately equal to the neutral rate of interest, where we define "neutral" as follows:

Definition 5 *The neutral rate of interest of an economy in which prices are sticky and the target rate is equal to the nominal interest rate, is the value of the equilibrium nominal rate of interest that prevails when output is equal to what it would be under flexible prices.*

To obtain impulse response functions, we segregate the paths that experience an up productivity shock at $t = 45$ from those that experience a down productivity shock at that time. We then compute conditional average paths for each of the two subsets of paths. We call "impulse-response function" the difference between the two conditional averages, normalized (thereby detrended) by the unconditional average path. Figure 4 shows the responses of output, the real stock market return index and the price level to a 2% productivity shock. Output and the price level take several years to reach new levels, increased by 2% and reduced by 3% respectively, while the real stock market return index reacts slightly more quickly with a 2% increase on impact.

5.2 The role of productivity shocks over several periods

As we saw in Sections 2 and 3, the rate of growth of ouput or total income, in the IID productivity-growth, binomial example with flexible prices, takes only two values. Based on the aggregate-demand model, output and the rate of inflation take two values, while the nominal rate of return on equity is equal to the constant rate of interest. Therefore, if prices set by firms were fully flexible, there would be no relationship whatever between the nominal return on stocks and the rate of inflation.

When the prices set by firms are sticky, however, output relative to flexible output depends on the previous-period price level, thus generating richer dynamics for stock returns and inflation. There can occur many values for inflation and many values for nominal stock returns depending on the value of the previous output. Over a single time-step, productivity can be up or down. When it is up, inflation is lower and nominal stock returns are higher than when it is down. However,

Observation 6 *Contingent upon productivity being up or down, for different values of current output relative to flexible output, the next-period realized inflation and the next-period realized nominal stock returns are near-linearly, positively related.*

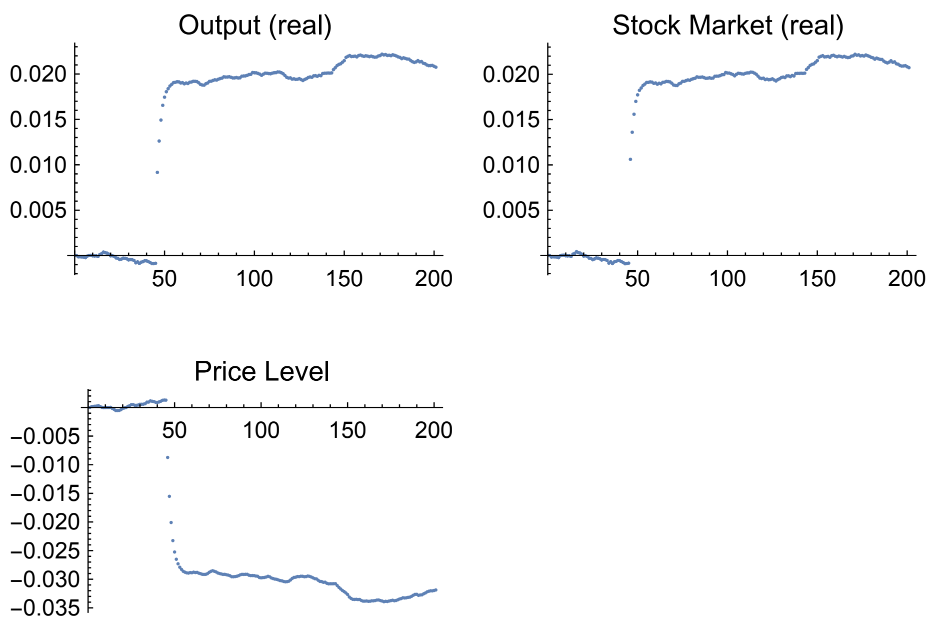

Figure 4: **Impulse responses: average path conditional on an "up" productivity shock occurring at $t = 45$ minus average path conditional on a "down" productivity shock occurring at that time.** Top left-hand panel: response of output (t in years on the x-axis). Top right-hand panel: response of the stock market (in real terms). Bottom panel: response of the price level**.** All responses are scaled by the corresponding unconditional average. Parameter values are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5). The figure is obtained from 10,000 paths drawing at random the event of an "up" or a "down" productivity shock z over 200 time steps of one year each.

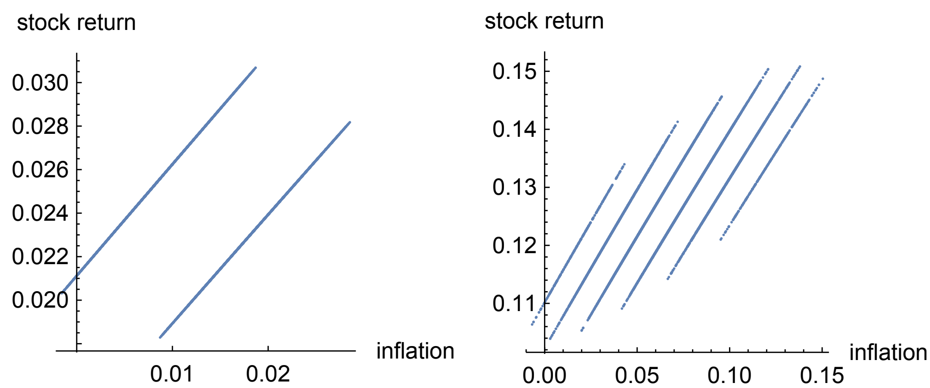

Figure 5: **Relation between one-period nominal stock return and one-period inflation (left-hand panel) and relation between the same two variables measured over five periods (right-hand panel)**, across 10,000 paths at a fixed date. Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5).

That pattern is illustrated in Figure 5, left-hand panel, which displays the two variables in a cross-section of paths.[26] In accordance with Observation 6, both the radii that appear are increasing near-straight lines. Of the two radii, the upper (lower) one portrays the relation conditional upon productivity being up (down). On a given radius, the points that plot farther to the North-East correspond to higher values of the ratio output/flexible-price output.[27] Across all the paths, not conditioning on the productivity growth, the coefficient of an across-paths regression of the nominal stock return on inflation between the two variables is equal to 0.1005.

When measuring returns over a longer holding period, the relationship is similar but more combinations of up and down productivity moves are possible. For instance, over five periods, six combinations are possible. The six corresponding radii are shown in Figure 5, right-hand panel: the top radius reflects realizations in which all five productivity moves are up while the bottom one reflects realizations in which they were all down. The radius second from the top contains those for which four moves where up and one down in any order etc. Across all the paths, not conditioning on the productivity growth combinations, the coefficient of an across-paths regression of the nominal stock return on inflation between the two variables is equal to 0.245. The interesting fact to observe is that the slope is higher over five periods than it is over one. This is

[26] When drawn along one path, the picture is near identical.

[27] That ratio appears to be bounded above and below.

Statistic	α_1	β_1	α_5	β_5
Median	0.023	0.097	0.111	0.243
Upper quintile	0.024	0.137	0.112	0.254
Lower quintile	0.022	0.049	0.11	0.23
Std error				
Median	0.00	0.015	0.001	0.014
Upper quintile	0.00	0.018	0.001	0.017
Lower quintile	0.00	0.012	0.001	0.011

Table 2: **Stock Returns and Contemporaneous Inflation:** the regressions are those of Equation (26). Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5). The table is obtained from 10,000 paths drawn at random.

because the six radii of five periods are more spread out than the two radii of one period.[28]

5.3 *Ex post* regression result

We ask whether the message conveyed by Figure 5 is confirmed by *time-series* regressions. Can the model fit the facts alluded to in Section 1? Having simulated 10,000, 200-period long paths of the economy of Section 4, we run on each an *ex post* regression in the manner of Boudoukh Richardson (1993) (BR).[29]

The *ex post* regression being run is quite simply:[30]

$$R_{t\to t+j} = \alpha_j + \beta_j \times \pi_{t\to t+j} + \varepsilon_{t,j} \tag{26}$$

where R is the nominal rate of return on the equity and π is the rate of inflation, with $j = 1$ for the one-year time interval and $j = 5$ for the five-year time interval. If the real rate of return on stock were constant, one would expect $\alpha_j = 0$ and $\beta_j = 1$. Because the five-year rates of return are calculated every year, there is overlap in the data and the Generalized Method of Moments is used to compute heteroskedasticity- (and autocorrelation-) consistent standard errors. The results are shown in Table 2.

[28] They are more spread out in the (inflation, stock return) plane because, except for the topmost and bottommost radii, all other radii are actually a bundle of radii that are partially superimposed and overlapping depending on the order in which the productivity shocks occur. In, for instance, the (inflation, stock return, current output relative to flexible output) space, they would be separate radii.

[29] We drop the first ten periods of the paths to ensure that statistical results do not depend on the initial condition, which is just the nominal amount $\bar{\theta}_2$ of government debt outstanding at $t = 0$.

[30] The exercise is descriptive. We are not testing a hypothesis and do not assume that inflation is an exogenous variable. Furthermore, if one wanted to hedge inflation risk using equities, one should calculate the hedge ratio (i.e., the number of units of stock to buy) by regressing inflation on stock returns.

Statistic	α_1	β_1	α_5	β_5
Median	0.017	0.519	0.106	0.309
Upper quintile	0.018	0.652	0.107	0.325
Lower quintile	0.015	0.436	0.105	0.298
Std error				
Median	0.001	0.059	0.001	0.010
Upper quintile	0.001	0.093	0.002	0.02
Lower quintile	0.001	0.04	0.001	0.013

Table 3: **Stock Returns and Expected Inflation: the Instrumental Variable Approach as in Equation (27).** Past output is the instrument. Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5). The table is obtained from 10,000 paths drawn at random.

The results are exactly in conformity with the intuition conveyed above, in that the five-year regression slope is much less negative than the one-year slope. The results are also in conformity with the empirical results of BR. Recall that both their slope coefficients were positive, but with the exact same disparity.[31]

Basically, therefore, we have discovered the reason for which BR found different slopes for different holding-period lengths.

5.4 *Ex ante* regression result

We test the moment conditions:

$$\mathbb{E}\left[\left(R_{t\to t+j} - \alpha_j - \beta_j \times \pi_{t\to t+j}\right) \otimes Z_t\right] = 0 \tag{27}$$

where Z_t is some set of instrumental variables known to investors at time t.

The exact *ex ante* formula that would correspond to the model is the CAPM (25) that applies to the equity. That CAPM in no way implies that the conditionally expected real rate of return on equity is constant. It is obviously a function of the state variable y_t, the current level of output. For that reason, we try one of the ex ante specifications of BR that involves the current level of output as the instrumental variable.

The results are shown in Table 3.

Recall that BR found one-year slope coefficients that were markedly smaller than the five-year coefficients. To the degree that the moments are conditional on the previous year's level of output, one may understand that the relationship between stock returns and inflation is restored to being positive. However, the comparative magnitudes of our simulated results for β_1 and β_5 in the *ex ante* specifications are the reverse of what BR found.

[31] Had we included in our model a monetary shock, we could also have increased both our simulated slope coefficients at will.

In our model, the previous year's level of output is a strong instrument. It is conceivable that, in the data, the instruments used by BR were not as strong.

6 Money demand and the zero lower bound

We now investigate the behavior of money demand and supply in the equilibrium of the last section. To do that, we build an equilibrium model of money demand along the lines of Baumol (1952) and Tobin (1956). We must observe at the outset that, when the nominal rate of interest approaches zero, money demand grows steadily thereby creating *a natural lower bound* on the rate of interest. Meanwhile, the demand of the private sector for the government bond drops steadily. Eventually the demands for money and bonds become indeterminate while their sum remains determinate and finite. The government cum central bank, as noted by J. M. Keynes, falls into a "liquidity-trap" regime that is akin to Quantitative Easing.[32]

Now there is cash explicitly in the economy, side by side with government bonds. Calling M monetary claims, money supply at time t is: $M_{2,t}$ (a negative number because, like $\theta_{2,t}$, it is a liability of the government cum central bank); money demand is $M_{1,t}$; the seignorage, an indirect tax, collected at time t and measured in nominal terms of that date is: $M_{1,t} \times (1 - 1/(1 + i_t))$. Households receive an income of a single good and no income in cash. At time t, the financial wealth available for consumption is:

$$P_t \times y_t + \theta_{1,t-1} + M_{1,t-1} - F_{1,t} - S_t$$

The proceeds $P_t \times y_t$ from the sale of the physical income are in the form of a deposit at a bank. Cash on hand $M_{1,t-1}$ and the other terms are assumed to be readily available in cash. Cash can be withdrawn by taking trips to the bank. Each trip costs a fixed real amount k. The smaller the number of trips $N_{1,t}$ the household decides to take to the bank, the more cash the household holds on an average over the time period $[t, t+1)$:[33]

$$M_{1,t} = \frac{P_t \times y_t}{2 \times N_{1,t}}$$

[32] On the zero lower bound, a very active topic of research during the Great Recession, see the following papers, which have implications for Finance: McCallum (2000), Krippner (2012), Wright (2012), Gavin et al. (2013), Priebsch (2013), Greenwood et al. (2014), Swanson and Williams (2014).

[33] We could have assumed that all the financial wealth except cash on hand is deposited with a bank. Then,

$$M_{1,t} = \frac{P_t \times y_t + \theta_{1,t-1} - F_{1,t} - S_t}{2 \times N_{1,t}}$$

so that the cost of the trips at current prices is:

$$k \times P_t \times N_{1,t} = k \times P_t \times \frac{P_t \times y_t + \theta_{1,t-1} - F_{1,t} - S_t}{2 \times M_{1,t}}$$

The derivation of the nodal system under that assumption is available in Appendix F.

so that the cost of the trips at current prices is:

$$k \times P_t \times N_{1,t} = k \times P_t \times \frac{P_t \times y_t}{2 \times M_{1,t}}$$

That cost is truly a deadweight loss; no one gets the benefit of it. But, for the sake of computational simplicity we imagine that it is refunded to the private sector in the form of a transfer $\zeta_{1,t} = P_t \times y_t \times k \times P_t / (2 \times M_{1,t})$, thus keeping in our equation system only the distortionary effect of the cost but not its wealth effect.[34] When money demand is optimized, the transfer amounts to a refund of the seignorage, as first-order condition (34) in the appendix indicates.[35] At the terminal point T, however, money is not "refunded." Even without a refund, the private sector holds it till the end because it has to. We set $1/(1+i_T) = 0$.

In Appendix E, we derive the set of equations (35) to be solved at each node of the tree. Eliminating the money terms from it and taking (34) into account:

Flow budget constraints of private sector

$$P_{t+1,j} \times c_{t+1,j} + F_{1,t+1,j} + \frac{P_{t+1,j} \times \sqrt{\frac{1}{2} y_{t+1,j} \times \frac{k}{1-\frac{1}{1+i_{t+1,j}}}}}{1+i_{t+1,j}} + S_{t+1,j}$$

$$= \theta_{1,t} + P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}} + P_{t+1,j} \times y_{t+1,j};$$

$$F_{1,T,j} = 0; j = u, d$$

Flow budget constraints of government cum central bank

$$F_{2,t+1,j} - \frac{P_{t+1,j} \times \sqrt{\frac{1}{2} y_{t+1,j} \times \frac{k}{1-\frac{1}{1+i_{t+1,j}}}}}{1+i_{t+1,j}}$$

$$= \theta_{2,t} - P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}} + S_{t+1,j}; F_{2,T,j} = 0; j = u, d$$

Portfolio-choice or Euler conditions

$$\frac{1}{1+i_t}\frac{1}{P_t} = \rho \frac{\frac{1}{2}\left(c_{t+1,u}\right)^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2}\left(c_{t+1,d}\right)^{\gamma-1} \frac{1}{P_{t+1,d}}}{\left(c_t\right)^{\gamma-1}}$$

Market clearing

$$\theta_{1,t} + \theta_{2,t} = 0$$

[34] Without that assumption, the trips to the bank being deadweight losses $c_t \neq y_t$.

[35] In addition, to preserve scale invariance (see footnote 22), we do not take k to be a constant; we assume it proportional to output.

Initial conditions are:

$$\begin{aligned} \frac{P_0}{P_{t-1}} &= Phill_0\left(y_0\right) \\ F_{2,0}\left(y_0\right) - \frac{P_0 \times \sqrt{\frac{1}{2} y_0 \times \frac{k}{1-\frac{1}{1+i_0}}}}{1+i_0} &= \theta_{2,-1} + M_{2,-1} + S_0 \end{aligned} \tag{28}$$

A change of unknown variables θ:

$$\hat{\theta}_{1,t} \triangleq \theta_{1,t} + P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}}; \quad \hat{\theta}_{2,t} \triangleq \theta_{2,t} - P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}}$$

along with a change of backward iterates:[36]

$$\hat{F}_{1,t} \triangleq F_{1,t} + \frac{P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}}}{1+i_t}; \quad \hat{F}_{2,t} \triangleq F_{2,t} - \frac{P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1-\frac{1}{1+i_t}}}}{1+i_t}$$

transforms the system of equations into one that is identical to the system (7), which we solved in the absence of money. We thus demonstrate that, *for a given value of the endogenous variable y_t*, money is simply added to government bonds and is *otherwise irrelevant*. The government surplus being exogenous anyway,[37] seignorage being refunded and inflation targeting being an infinitely elastic central-bank reaction function, money demand only serves to determine money supply, as has been pointed out by many authors.

This is true with two caveats. Firstly, the change of variables is valid only for strictly positive nominal interest rates. If we implemented it blindly, the nominal rate of interest could become negative, unwarrantedly so. To prevent that error, we replace the natural lower bound by an artificial one. We superimpose on the Taylor rule a zero lower bound on the nominal rate of interest:[38]

$$1 + i_t = \max\left[1, (1+\bar{\imath}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u} + \frac{1}{2}P_{t+1,d}}{P_t}}{1+\bar{\pi}}\right)^{\phi}\right] \tag{29}$$

Secondly, since we have assumed that money is not refunded, the terminal conditions, which were originally $F_{1,T,j} = F_{2,T,j} = 0$ must be replaced by: $\hat{F}_{1,T,j} = -\hat{F}_{2,T,j} = P_T \times \sqrt{\frac{1}{2} y_T \times k}$. We intend to study the paths of the economy in a stationary situation. The change of terminal condition is not very important except for the fact that it modifies the solution to the initial conditions (28), so that the initial price level P_0 and the initial output y_0 *are affected by the presence of money.* The initial point being modified, every path of the economy

[36] Note: $\hat{\theta}_{1,t} / (1+i_t) = \hat{F}_{1,t}$

[37] But see below the caveat concerning the terminal condition.

[38] We actually implement a smooth variant of that relation.

will also be modified but the dynamics of the system will not, unless the nominal rate of interest approaches the zero lower bound.

We amend the "aggregate demand" subsystem of equations of Section 2 to reflect the modified policy rule (29), leaving intact the "aggregate supply" subsystem of Section 4 and we solve by backward induction exactly as we did before (with the additional parameter $k = 1\%$ of output). Under the parameter and state variable combinations considered so far, the result is identical to that of Figure 3, simply because the cashless economy itself never produced a negative value for the rate of interest. In order to make liquidity-trap episodes possible, we reduce the target rate of interest of the Taylor rule by 1% below the neutral rate of the monetary economy.

We now discuss the outcome of that experiment.

The new version of Figure 5 is Figure 6, which shows that the lower bound on the rate of interest introduces a support from below for realized nominal stock returns. For that reason, the relation between inflation and stock returns described above in section 5.2 is no longer near linear but is still positive, contingent on a given sequence of productivity shocks. Not conditioning on the productivity growth, the coefficient of an across-paths regression of the nominal stock return on inflation between the two variables is equal to 0.0743 over one period while it is equal to 0.1950 over five periods. Once again the slope is quite a bit larger over five periods than it is over one.

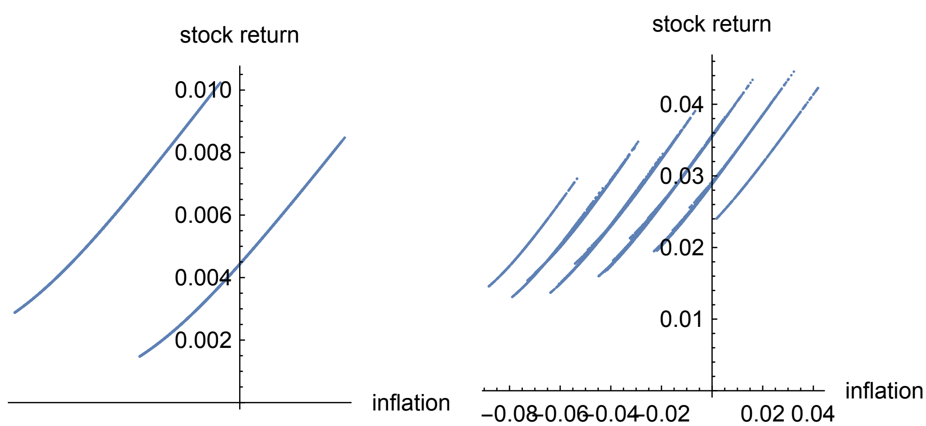

Figure 6: **Relation, in the presence of money, between one-period nominal stock return and one-period inflation (left-hand panel) and relation between the same two variables measured over five periods (right-hand panel)**, across 10,000 paths at a fixed date. Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5).

Statistic	α_1	β_1	α_5	β_5
Median	0.006	0.072	0.031	0.195
Upper quintile	0.006	0.104	0.032	0.207
Lower quintile	0.005	0.033	0.031	0.182
Std error				
Median	0.00	0.012	0.001	0.011
Upper quintile	0.00	0.014	0.001	0.014
Lower quintile	0.00	0.01	0.00	0.009

Table 4: **Stock Returns and Contemporaneous Inflation in the presence of money:** the regressions are those of Equation (26). Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5). The table is obtained from 10,000 paths drawn at random.

The new version of Table 2, which contained the results of *ex post* regressions across simulated paths, is Table 4.

The results are again in conformity with the empirical results of BR. We do not display the *ex ante* regression results, which are once again ambiguous.

7 Bonds and the 'Fed model'

Asness (2003) criticizes a heuristic approach of professional circles who compare yields on stock securities to yields on bonds, and expect the two to revert to each other, which, empirically speaking, they do, both yields being high when inflation is high.[39] He refers to this approach as the "Fed model". He points out correctly that the two yields are not comparable; the coupon payments on a bond are constant in current euros while the dividends on a share of stock will grow with inflation.

This attitude of professional circles may be a form of money illusion, reminiscent of Modigliani and Cohn (1979), unless one of two rational explanations of the fact that the "Fed model" works well empirically holds. One is the hypothesis that says that high current inflation is associated with *low* expected long-run nominal dividend growth in excess of the riskless rate, justifiably driving up the equity dividend yield. The other is that high inflation drives up the risk of the economy and thus the nominal equity risk premium. Campbell and Vuolteenaho (2004) (CV) ran an empirical investigation to determine which of the three possibilities transpires in the behavior of stock prices. They decompose the dividend yield on equity into three components: a constant, expected long-run future nominal equity excess returns and expected long-run nominal dividend growth in excess of the riskless rate. They find that high current inflation is associated

[39] Maio (2013) shows empirically that show that the yield gap forecasts excess market returns, both at short and long forecasting horizons, and for both value- and equal-weighted stock indexes, and it also outperforms competing predictors commonly used in the literature.

with *high* expected long-run nominal dividend growth in excess of the riskless rate, and that inflation is not related to the anticipated nominal equity premium, thus leaving money illusion as the surviving hypothesis.

Indirect empirical evidence of the effect of monetary policy on the stock market is also provided by a recent paper of David and Veronesi (2013) relating stock returns to returns on bonds. The model allows for money illusion on the part of investors. The authors argue that realized inflation is interpreted very differently by investors depending on whether they fear stagflation (as in the 1980's) – a fear that leads to a high correlation – or deflation, which would lead to a lower correlation.

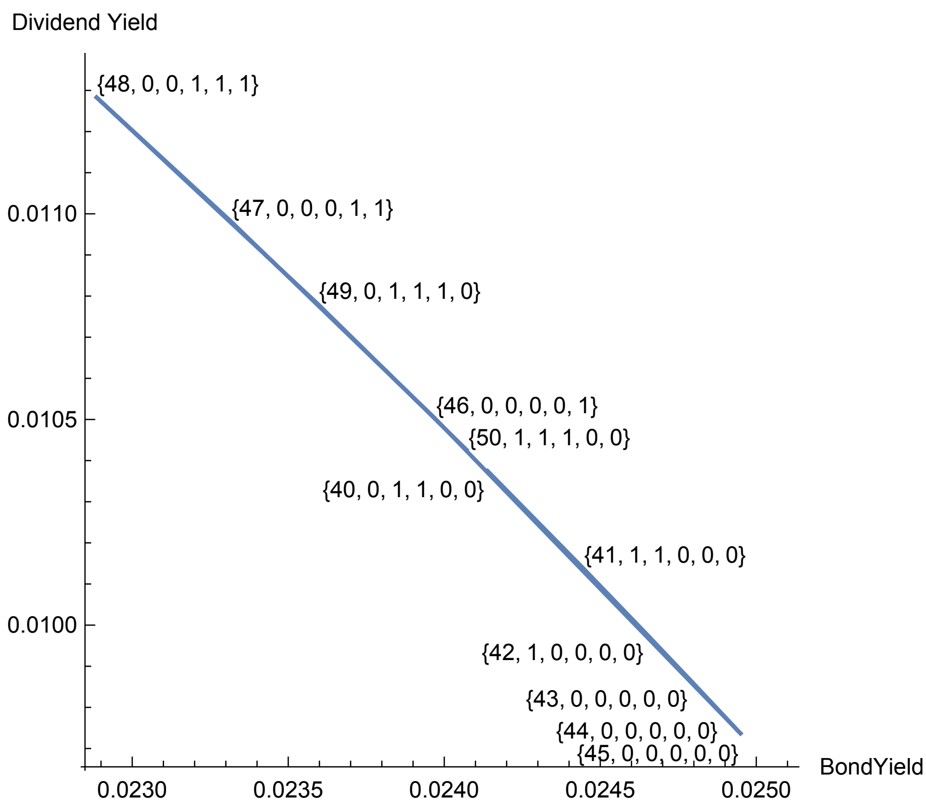

Figure 7: **Relation between nominal (or real) dividend yield on equity and nominal annualized yield on a ten-year nominal bond**, across 10,000 paths at a fixed date. The labeling of the points of dates 40 to 50 along one example path contains the following information: {date (year t), up or down productivity shock (coded 1 and 0) two years ago, one year ago and contemporaneously, and the contemporaneous rate of inflation}. Parameters are as in Table 1 except for the target rate of interest, which is set approximately at a neutral level (see Definition 5).

To examine these issues, we now revert to the cashless economy set up (with the parameter values of Table 1) and introduce a ten-year zero-coupon bond that pays one current monetary unit at maturity, just like the stock pays dividends forever. We find (Figure 7) that there exists a near-straight line *negatively sloped* relationship between dividend yield and bond yield. The labelling of a few of the simulated points tell the dynamic story: as negative productivity shocks accumulate, inflation becomes higher and higher, driving up the bond yield while the dividend yield is brought down by the negative productivity shocks. Dividend yield and bond yield do not move in tango and should, therefore, not be compared.

The theoretical result is not consistent the empirical evidence metnioned in the opening paragraphs of this section. This evidence, however, is entirely based on post-World War II data, which feature a long upward swing of yields to a peak in the early 80s followed by a long downward swing of both yields. The up and down swings are probably caused the change of monetary policy regime that took place around 1980. If one extends the sample to the nineteenth century, one finds that the empirical relationship between yields no longer holds.

8 Conclusion

Adopting a method that has been used to calculate dynamic financial-market equilibria, we have constructed the equilibrium of a cashless production economy with productivity shocks and with three types of agents: (i) household/investors who supply labor with a finite elasticity, consume a large variety of goods that are not perfect substitutes and trade government bonds; (ii) firms that produce those varieties of goods, setting prices in a Calvo manner; (iii) a government that collects an exogenous fiscal surplus and acts mechanically, buying and selling bonds in accordance with a Taylor policy rule based on expected inflation. Merging the consumption-financial behavior of households with the policy rule, we have derived explicitly at each point in time and in each state of nature, aggregate-demand schedules (inclusive of policy rule) relating, at the next point in time and in each successor state, the price level to the level of output. For a short horizon, we have shown that these schedules are decreasing if and only if the exponent of the Taylor rule that falls on expected inflation is less than 1. The aggregate supply schedules (or Phillips curve) that also apply to the next point in time are always increasing. The equilibrium is unique if the exponent is less than 1. Otherwise, because of the non linearities of the two types of schedules, two equilibria can exist.

In this equilibrium, we have priced the stock market, defined as the present discounted value of firms' profits and simulated the joint behavior of stock returns and inflation. That has allowed us to discover the reason for which Boudoukh Richardson (1993) found different slopes for different holding-period lengths. The reason lies in the succession of productivity shocks that take place over several periods, which is a newfangled version of Fama (1981)'s "proxy hypothesis" explained in the introduction.

The equilibrium has then been expanded to incorporate an explicit money demand à la Baumol and Tobin. The only effect of the zero lower bound thus created as been to support stock returns when they are low.

Finally, we examined the validity of the 'Fed model' in the context of our model. The model invalidates completely the suggestion that one might compare dividend yields to bond yields to assess the direction of the stock market. The relationship between them is in fact negative: when inflation is high, the bond yield is high while the dividend yield is low.

Appendixes

A The backward equation system for the non-Ricardian, real-surplus case of subsection 2.1

The dynamic programming formulation of the investor's problem is:

$$J_1\left(\theta_{1,t-1},\cdot,t\right)=\sup_{c_t,\{\theta_{1,t,i}\}} u\left(c_t,t\right)+\mathbb{E}_t J_1\left(\theta_{1,t},\cdot,\widetilde{y}_{t+1},t+1\right) \tag{30}$$

subject to the flow budget constraint written at time t only.

The Lagrangian for problem (30) is:

$$\begin{aligned}\mathcal{L}_1\left(\theta_{1,t-1,i},\cdot,t\right)=&\sup_{c_t,\theta_{1,t}}\inf_{\phi_{1,t}} u\left(c_t,t\right)\\&+\sum_{j=u,d}\pi_{t,t+1,j}J_1\left(\theta_{1,t},\cdot,t+1\right)\\&+\phi_{1,t}\times\left[\theta_{1,t-1}+P_t\times y_t-P_t\times c_t-\frac{\theta_{1,t}}{1+i_t}-s_t\times P_t\right]\end{aligned}$$

where $\phi_{1,t}$ is the Lagrange multiplier attached to the flow budget constraint (3). The first-order conditions are:

$$\begin{aligned}u'\left(c_t,t\right)&=\phi_{1,t}\times P_t\\\theta_{1,t-1}+P_t\times y_t-P_t\times c_t-\frac{\theta_{1,t}}{1+i_t}-s_t\times P_t&=0\\\sum_{j=u,d}\pi_{t,t+1,j}\frac{\partial J_{1,t+1,j}}{\partial\theta_{1,t,i}}\left(\theta_{1,t},\cdot,t+1\right)&\\=\phi_{1,t}\times\frac{1}{1+i_t}&\end{aligned} \tag{31}$$

In order to eliminate the value function from the first-order conditions, we differentiate the Lagrangian with respect to $\theta_{1,t-1,i}$:

$$\frac{\partial J_1}{\partial\theta_{1,t-1,i}}=\frac{\partial L_1}{\partial\theta_{1,t-1,i}}=\phi_{1,t}$$

so that the first-order conditions can also be written:

$$\begin{aligned}u'\left(c_t,t\right)&=\phi_{1,t}\times P_t\\\theta_{1,t-1}+P_t\times y_t-P_t\times c_t-\frac{\theta_{1,t}}{1+i_t}-s_t\times P_t&=0\\\sum_{j=u,d}\pi_{t,t+1,j}\phi_{1,t+1,j}&=\phi_{1,t}\times\frac{1}{1+i_t}\end{aligned} \tag{32}$$

As has been noted by Dumas and Lyasoff (2012) in a different context, the system made of (32) and (6) above has a drawback. It must be solved simultaneously (or globally) for all nodes of all times. As written, it cannot be solved recursively in the backward way because the unknowns at time t include consumptions at time t, c_t, whereas the third subset of equations in (32) if rewritten as:

$$\sum_{j=u,d} \pi_{t,t+1,j} \times \frac{u'(c_{t+1,j}, t)}{P_{t+1,j}} = \phi_{1,t} \times \frac{1}{1+i_t}$$

can be seen to be a restriction on consumptions at time $t+1$, which at time t would already be solved for.

In order to “synchronize” the solution algorithm of the equations and allow recursivity, we first shift all first-order conditions, except the third one, forward in time and, second, we no longer make explicit use of the investor’s position $\theta_{1,t-1}$ held when entering time t, focusing instead on the financial wealth: $F_{1,t} \triangleq \frac{\theta_{1,t}}{1+i_t}$ held when exiting time $t+1$, which are carried backward. Regrouping equations in that way leads to the equation system of Section 2.1.

B Proof of equation (21)

The first-order condition is:

$$\sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \left((1-\sigma) \frac{P_{v,t}^{-\sigma}}{P_{t+i}^{1-\sigma}} + \sigma \varphi_{t+i} \frac{P_{v,t}^{-\sigma-1}}{P_{t+i}^{-\sigma}} \right) y_{t+i} \right] = 0$$

Divide by $P_{v,t}^{-\sigma}$:

$$\sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \left((1-\sigma) \frac{1}{P_{t+i}^{1-\sigma}} + \sigma \varphi_{t+i} \frac{P_{v,t}^{-1}}{P_{t+i}^{-\sigma}} \right) y_{t+i} \right] = 0$$

Split:

$$(1-\sigma) \sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \frac{1}{P_{t+i}^{1-\sigma}} y_{t+i} \right] + \sigma P_{v,t}^{-1} \sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \varphi_{t+i} \frac{1}{P_{t+i}^{-\sigma}} y_{t+i} \right] = 0$$

Multiply by $P_t^{1-\sigma}$:

$$(1-\sigma) \sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \frac{P_t^{1-\sigma}}{P_{t+i}^{1-\sigma}} y_{t+i} \right] + \sigma P_t P_{v,t}^{-1} \sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \varphi_{t+i} \frac{P_t^{-\sigma}}{P_{t+i}^{-\sigma}} y_{t+i} \right] = 0$$

Solving for $P_t P_{v,t}^{-1}$ gives (21).

C Approximation

In a neighborhood of zero inflation, one might be willing to make the approximation:

$$\left[\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{-\sigma} + (1-\omega) \times \left(\frac{1-\omega \times \left(\frac{P_{t-1}}{P_t}\right)^{1-\sigma}}{1-\omega}\right)^{-\frac{\sigma}{1-\sigma}}\right] \simeq 1 \qquad (33)$$

in which case one can get the Phillips curve explicitly:

$$\frac{P_t}{P_{t-1}} = \left[\frac{\omega}{1-(1-\omega)\left(\frac{\sigma}{\sigma-1}\frac{\left(\frac{y_t}{z_t}\right)^{\eta}+A_{approx}(t,y_t)}{y_t^{\gamma}+B_{approx}(t,y_t)}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}$$

where:

$$\begin{aligned} A_{approx}\left(t,y_t\right) &= \rho\omega\mathbb{E}_t\left(\frac{P_{t+1}}{P_t}\right)^{\sigma}\left[\left(\frac{y_{t+1}}{z_{t+1}}\right)^{\eta} + A_{approx}\left(t+1,y_{t+1}\right)\right] \\ B_{approx}\left(t,y_t\right) &= \rho\omega\mathbb{E}_t\left(\frac{P_{t+1}}{P_t}\right)^{\sigma-1}\left[y_{t+1}^{\gamma} + B_{approx}\left(t+1,y_{t+1}\right)\right] \end{aligned}$$

D Walras' law in the sticky-price system

Aggregating the budget constraints in each state ($j = u, d$):

$$P_{t+1,j} \times c_{t+1,j} + F_{1,t+1,j} + F_{2,t+1,j} + s_{t+1,j} \times P_{t+1,j}$$

$$= \theta_{1,t} + \theta_{2,t} + s_{t+1,j} \times P_{t+1,j} + P_{t+1,j} \times \delta_{t+1,j} + W_{t+1,j} \times l_{t+1,j};\ F_{1,T,j} = 0$$

$$P_{t+1,j} \times c_{t+1,j} = P_{t+1,j} \times \delta_{t+1,j} + W_{t+1,j} \times l_{t+1,j}$$

$$P_{t+1,j} \times c_{t+1,j} = P_{t+1,j} \times \left[\omega \times \left(\frac{P_t}{P_{t+1,j}} - \varphi_{t+1,j}\right)\left(\frac{P_t}{P_{t+1,j}}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_{t+1}^*}{P_{t+1,j}} - \varphi_{t+1,j}\right)\left(\frac{P_{t+1,j}^*}{P_{t+1,j}}\right)^{-\sigma}\right] \times y_{t+1,j} + W_{t+1,j} \times l_{t+1,j}$$

$$P_{t+1,j} \times c_{t+1,j} = P_{t+1,j} \times \left[\omega \times \left(\frac{P_t}{P_{t+1,j}} - \frac{W_{t+1,j}}{z_{t+1,j} \times P_{t+1,j}}\right)\left(\frac{P_t}{P_{t+1,j}}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_{t+1}^*}{P_{t+1,j}} - \frac{W_{t+1,j}}{z_{t+1,j} \times P_{t+1,j}}\right)\left(\frac{P_{t+1,j}^*}{P_{t+1,j}}\right)^{-\sigma}\right] \times y_{t+1,j} + W_{t+1,j} \times l_{t+1,j}$$

$$P_{t+1,j} \times c_{t+1,j} = P_{t+1,j} \times \left[\omega \times \frac{P_t}{P_{t+1,j}}\left(\frac{P_t}{P_{t+1,j}}\right)^{-\sigma} + (1-\omega) \times \frac{P_{t+1}^*}{P_{t+1,j}}\left(\frac{P_{t+1,j}^*}{P_{t+1,j}}\right)^{-\sigma}\right] \times y_{t+1,j}$$

$$+\left(-\frac{W_{t+1,j}}{z_{t+1,j}}\right) \times \left[\omega \times \left(\frac{P_t}{P_{t+1,j}}\right)^{-\sigma} + (1-\omega) \times \left(\frac{P_{t+1,j}^*}{P_{t+1,j}}\right)^{-\sigma}\right] \times y_{t+1,j} + W_{t+1,j} \times l_{t+1,j}$$

Cancellation produced by (23) gives:

$$P_{t+1,j} \times c_{t+1,j} = P_{t+1,j} \times \left[\omega \times \frac{P_t}{P_{t+1,j}} \left(\frac{P_t}{P_{t+1,j}} \right)^{-\sigma} + (1-\omega) \times \frac{P^*_{t+1}}{P_{t+1,j}} \left(\frac{P^*_{t+1,j}}{P_{t+1,j}} \right)^{-\sigma} \right] \times y_{t+1,j}$$

$$c_{t+1,j} = \left[\omega \times \left(\frac{P_t}{P_{t+1,j}} \right)^{1-\sigma} + (1-\omega) \times \left(\frac{P^*_{t+1,j}}{P_{t+1,j}} \right)^{1-\sigma} \right] \times y_{t+1,j}$$

$$c_{t+1,j} = y_{t+1,j}$$

Therefore, all accounts are straight: clearing of the financial market and of the labor market do imply clearing of the goods market.

E Backward equation system for the Baumol-Tobin case

In the entire paper, W_t is the nominal wage rate. In this appendix only, the symbol W stands for entering (or pre-trade) wealth.

$$\mathcal{L}_1 \left(W_{1,t}, \cdot, t \right) = \sup_{c_t, \theta_{1,t}} \inf_{\phi_{1,t}} u_1 \left(c_t, t \right)$$

$$+ \sum_{j=u,d} \pi_{t,t+1,j} J_1 \left(\theta_{1,t}, M_{1,t}, \cdot, t+1 \right)$$

$$+\phi_{1,t} \times \left[W_{1,t} - \frac{\theta_{1,t}}{1+i_t} - S_t + P_t \times y_t \times \left(1 - \frac{k \times P_t}{2 \times M_{1,t}} \right) - P_t \times c_t - M_{1,t} + \zeta_{1,t} \right]$$

where: $W_{1,t} \triangleq M_{1,t-1} + \theta_{1,t-1}$. First-order condition with respect to $\theta_{1,t}$:

$$\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial}{\partial W_{1,t}} J_1 \left(\theta_{1,t}, M_{1,t}, \cdot, t+1 \right) - \frac{\phi_{1,t}}{1+i_t} = 0$$

First-order condition with respect to $M_{1,t}$:

$$\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial}{\partial W_{1,t}} J_1 \left(\theta_{1,t}, M_{1,t}, \cdot, t+1 \right) + \phi_{1,t} \times \left[P_t \times y_t \times \frac{k \times P_t}{2 \times \left(M_{1,t} \right)^2} - 1 \right] = 0$$

Envelope condition:

$$\frac{\partial}{\partial W_{1,t-1}} J_1 \left(\theta_{1,t-1}, M_{1,t-1}, \cdot, t \right) = \phi_{1,t}$$

The Euler conditions are:

$$\sum_{j=u,d} \pi_{t,t+1,j} \phi_{1,t+1,j} - \frac{\phi_{1,t}}{1+i_t} = 0; t = 0, ..., T-1$$

$$\sum_{j=u,d} \pi_{t,t+1,j} \phi_{1,t+1,j} = \phi_{1,t} \times \left[1 - P_t \times y_t \times \frac{k \times P_t}{2 \times \left(M_{1,t} \right)^2} \right]; t = 0, ..., T$$

The latter is simply:

$$\begin{aligned}
\frac{1}{1+i_t} &= 1 - P_t \times y_t \times \frac{k \times P_t}{2 \times (M_{1,t})^2} \\
P_t \times y_t \times \frac{k \times P_t}{2 \times (M_{1,t})^2} &= 1 - \frac{1}{1+i_t} \\
P_t \times y_t \times \frac{k \times P_t}{1 - \frac{1}{1+i_t}} &= 2 \times (M_{1,t})^2 \\
M_{1,t} &= P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1 - \frac{1}{1+i_t}}}
\end{aligned} \tag{34}$$

except at time $t = T$ where:

$$\begin{aligned}
1 - P_T \times y_T \times \frac{k \times P_T}{2 \times (M_{1,T})^2} &= 0 \\
M_{1,T} &= P_T \times \sqrt{\frac{1}{2} y_T \times k}
\end{aligned}$$

Summing up, the set of equations to be solved at each node of the tree is:

$$\begin{gathered}
\text{Flow budget constraints of private sector} \\
P_{t+1,j} \times c_{t+1,j} + F_{1,t+1,j} + M_{1,t+1,j} + S_{t+1,j} \\
= \theta_{1,t} + M_{1,t} + P_{t+1,j} \times y_{t+1,j} + \zeta_{1,t+1,j}; \\
F_{1,T,j} = 0; j = u, d \\
\text{Flow budget constraints of government cum central bank} \\
F_{2,t+1,j} + M_{2,t+1,j} = \theta_{2,t} + M_{2,t} + S_{t+1,j} - \zeta_{1,t+1,j}; F_{2,T,j} = 0; j = u, d \\
\text{Portfolio-choice or Euler conditions} \\
\frac{1}{1+i_t} \frac{1}{P_t} = \rho \frac{\frac{1}{2} (c_{t+1,u})^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2} (c_{t+1,d})^{\gamma-1} \frac{1}{P_{t+1,d}}}{(c_t)^{\gamma-1}}; t = 0, ..., T-1 \\
M_{1,t} = P_t \times \sqrt{\frac{1}{2} y_t \times \frac{k}{1 - \frac{1}{1+i_t}}}; t = 0, ..., T-1; M_{1,T} = P_T \times \sqrt{\frac{1}{2} y_T \times k} \\
\text{Market clearing} \\
\theta_{1,t} + \theta_{2,t} = 0; M_{1,t+1,u} + M_{2,t+1,u} = 0; M_{1,t+1,d} + M_{2,t+1,d} = 0; M_{1,t} + M_{2,t} = 0
\end{gathered} \tag{35}$$

F Backward equation system for the Baumol-Tobin case under alternative specification

$$\mathcal{L}_1\left(\theta_{1,t-1,i}, M_{1,t-1}, \cdot, t\right) = \sup_{c_t,\theta_{1,t}} \inf_{\phi_{1,t}} u_1\left(c_t, t\right)$$
$$+ \sum_{j=u,d} \pi_{t,t+1,j} J_1\left(\theta_{1,t}, M_{1,t}, \cdot, t+1\right)$$
$$+\phi_{1,t} \times \left\{M_{1,t-1} + \left[P_t \times y_t + \theta_{1,t-1} - \frac{\theta_{1,t}}{1+i_t} - S_t\right] \times \left(1 - \frac{k \times P_t}{2 \times M_{1,t}}\right) - P_t \times c_t - M_{1,t} + \zeta_{1,t}\right\}$$

First-order condition with respect to $\theta_{1,t}$:

$$\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial}{\partial \theta_{1,t}} J_1\left(\theta_{1,t}, M_{1,t}, \cdot, t+1\right) - \frac{\phi_{1,t}}{1+i_t} \times \left(1 - \frac{k \times P_t}{2 \times M_{1,t}}\right) = 0$$

First-order condition with respect to $M_{1,t}$:

$$\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial}{\partial M_{1,t}} J_1\left(\theta_{1,t}, M_{1,t}, \cdot, t+1\right) + \phi_{1,t} \times \left\{\left[P_t \times y_t + \theta_{1,t-1} - \frac{\theta_{1,t}}{1+i_t} - S_t\right] \times \frac{k \times P_t}{2 \times \left(M_{1,t}\right)^2} - 1\right\} = 0$$

Envelope conditions:

$$\begin{aligned}
\frac{\partial}{\partial \theta_{1,t-1}} J_1\left(\theta_{1,t-1}, M_{1,t-1}, \cdot, t\right) &= \phi_{1,t} \times \left(1 - \frac{k \times P_t}{2 \times M_{1,t}}\right) \\
\frac{\partial}{\partial M_{1,t-1}} J_1\left(\theta_{1,t-1}, M_{1,t-1}, \cdot, t\right) &= \phi_{1,t}
\end{aligned}$$

so that the Euler conditions are:

$$\begin{aligned}
\sum_{j=u,d} \pi_{t,t+1,j} \phi_{1,t+1,j} \times \left(1 - \frac{k \times P_{t+1,j}}{2 \times M_{1,t+1,j}}\right) - \frac{\phi_{1,t}}{1+i_t} \times \left(1 - \frac{k \times P_t}{2 \times M_{1,t}}\right) &= 0 \\
\sum_{j=u,d} \pi_{t,t+1,j} \phi_{1,t+1,j} + \phi_{1,t} \times \left\{\left[P_t \times y_t + \theta_{1,t-1} - \frac{\theta_{1,t}}{1+i_t} - S_t\right] \times \frac{k \times P_t}{2 \times \left(M_{1,t}\right)^2} - 1\right\} &= 0
\end{aligned}$$

The nodal system follows.

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